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An Investigation of the Limits of Geometric Constructions

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Abstract: We investigate the intersection of a line and its limit as two points defining the line coalesce. In this paper, the "limit line" is the line formed by two points as they coalesce.

Introduction – A Limit Point:

In this paper, we will examine the behavior of a line formed by two points as the points coalesce; in particular, we examine its behavior in regards to its intersection with some line M. Assume that when these points coalesce, a limit line L is formed. The intersection between line L and some other line is simply the intersection of lines L and M. What this paper examines is the intersection of line L with line M when L=M. Here is an example to clarify:

In the example, E is the intersection of BD with the x-axis. We examine the x-coordinate of E as r tends to 0

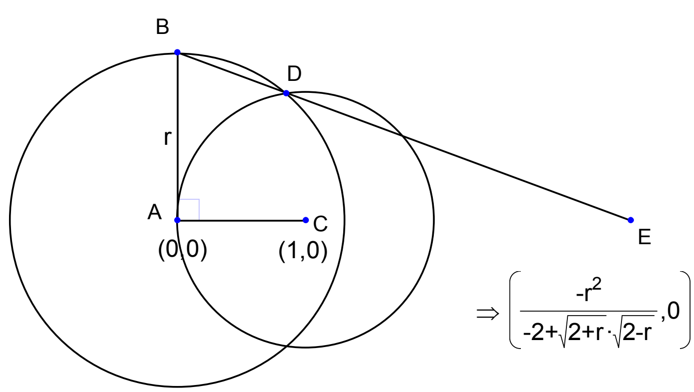

Figure 1: As r tends to 0, lines AE and BE coalesce, but their intersection remains well defined and finite

If we take the limit as r approaches 0 of the x-coordinate of point E, we see that the x-coordinate approaches 4. The purpose of this paper is to generalize this example. If we examine this example, we see that as r approaches 0, points B and D move towards the origin; when r is 0, they coalesce. The slope of the line BD approaches 0 as r approaches 0; therefore, the x-axis is the limit line of BD. This example is just a specific case of the intersection of a line and its limit line when the two points defining the line coalesce.

The Intersection of a Line and its Limit Line:

We now will examine the generalized case of the previous example, where the motion of the two points as well as the slope of the limit line can vary.

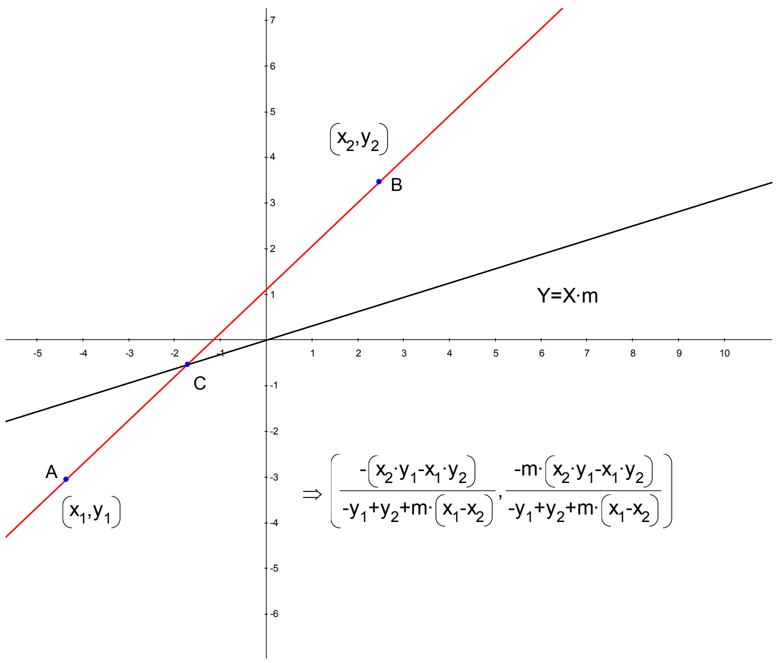

Figure 2: Generalized line intersection

Let points A and B be defined parametrically.

Point A: $X = x_1(t), Y = y_1(t)$

Point B: $X = x_2(t), Y = y_2(t)$

Such that the derivatives of the y-parameters are equal up to *n-1* derivations and the derivatives of the x-parameters are equal up to *n-1* derivations when they meet at the origin:

$$x_1(0) = y_1(0) = x_2(0) = y_2(0) = 0$$

$$y_1'(0) = y_2'(0),\ y_1''(0) = y_2''(0),\ \ldots,\ y_1^{n-1}(0) = y_2^{n-1}(0)$$

$$x_1'(0) = x_2'(0),\ x_1''(0) = x_2''(0),\ \ldots,\ x_1^{n-1}(0) = x_2^{n-1}(0)$$

As t approaches 0, the x- and y-coordinates of the intersection of the line AB with its "limit line" (the line formed when A and B coalesce) are determined by taking the limit of the equations for the X- and Y-coordinates, giving us the functions X(n) and Y(n):

X(n): $$\lim_{t\to 0}\left(\frac{y_1(t)x_2(t)-y_2(t)x_1(t)}{y_1(t)-y_2(t)-mx_1(t)+mx_2(t)}\right)=$$

$$\frac{(n+1)(y_1^{n}(0)x_2'(0)+x_2^{n}(0)y_1'(0)-y_2^{n}(0)x_1'(0)-x_1^{n}(0)y_2'(0))+y_1^{n+1}(0)x_2(0)+x_2^{n+1}(0)y_1(0)-y_2^{n+1}(0)x_1(0)-x_1^{n+1}(0)y_2(0)}{y_1^{n+1}(0)-y_2^{n+1}(0)-mx_1^{n+1}(0)+mx_2^{n+1}(0)}$$

Y(n): $$\lim_{t\to 0}\left(m\left(\frac{y_1(t)x_2(t)-y_2(t)x_1(t)}{y_1(t)-y_2(t)-mx_1(t)+mx_2(t)}\right)\right)=$$

$$m\left(\frac{(n+1)(y_1^{n}(0)x_2'(0)+x_2^{n}(0)y_1'(0)-y_2^{n}(0)x_1'(0)-x_1^{n}(0)y_2'(0))+y_1^{n+1}(0)x_2(0)+x_2^{n+1}(0)y_1(0)-y_2^{n+1}(0)x_1(0)-x_1^{n+1}(0)y_2(0)}{y_1^{n+1}(0)-y_2^{n+1}(0)-mx_1^{n+1}(0)+mx_2^{n+1}(0)}\right)$$

Where m is the slope of the limit line of AB

$$m=\lim_{t\to 0}\left(\frac{y_1(t)-y_2(t)}{x_1(t)-x_2(t)}\right)=\frac{y_1^{n}(t)-y_2^{n}(t)}{x_1^{n}(t)-x_2^{n}(t)}$$

We will prove this inductively.

The base case where n=2:

$$x_1(0)=y_1(0)=x_2(0)=y_2(0)=0$$
$$y_1'(0)=y_2'(0)$$
$$x_1'(0)=x_2'(0)$$

In this case the x- and y-parameters have equal first derivatives. Using these conditions, we take the limit of the equation of the X-and Y-coordinates:

$$\lim_{t\to 0}\left(\frac{y_1(t)x_2(t)-y_2(t)x_1(t)}{y_1(t)-y_2(t)-mx_1(t)+mx_2(t)}\right)=$$

$$\frac{3(y_1''(0)x_2'(0)+x_2''(0)y_1'(0)-y_2''(0)x_1'(0)-x_1''(0)y_2'(0))}{y_1'''(0)-y_2'''(0)-mx_1'''(0)+mx_2'''(0)}$$

Which is equal to X(2):

$$\frac{3(y_1''(0)x_2'(0)+x_2''(0)y_1'(0)-y_2''(0)x_1'(0)-x_1''(0)y_2'(0))}{y_1'''(0)-y_2'''(0)-mx_1'''(0)+mx_2'''(0)}$$

Y-Coordinate:

$$\lim_{t\to 0}\left(m\frac{y_1(t)x_2(t)-y_2(t)x_1(t)}{y_1(t)-y_2(t)-mx_1(t)+mx_2(t)}\right)=$$

$$m(\frac{3(y_1''(0)x_2'(0)+x_2''(0)y_1'(0)-y_2''(0)x_1'(0)-x_1''(0)y_2'(0))}{y_1'''(0)-y_2'''(0)-mx_1'''(0)+mx_2'''(0)})$$

Which is equal to Y(2):

$$m(\frac{3(y_1''(0)x_2'(0)+x_2''(0)y_1'(0)-y_2''(0)x_1'(0)-x_1''(0)y_2'(0))}{y_1'''(0)-y_2'''(0)-mx_1'''(0)+mx_2'''(0)})$$

X(n) and Y(n) hold for the base case where n=2. Now we will prove that X(n) and Y(n) hold for all n>2. Assuming that X(k) and Y(k) are true, we will prove the case of X(k+1) and Y(k+1):

$$x_1(0)=y_1(0)=x_2(0)=y_2(0)=0$$

$$y_1'(0)=y_2'(0),\ y_1''(0)=y_2''(0),\ \ldots,\ y_1^k(0)=y_2^k(0)$$

$$x_1'(0)=x_2'(0),\ x_1''(0)=x_2''(0),\ \ldots,\ x_1^k(0)=x_2^k(0)$$

X(k) and Y(k) are now equal to 0/0, so we apply l'Hopital's rule:

$$\lim_{t\to 0} X(k)=$$

$$\frac{(k+2)(y_1^{k+1}(0)x_2'(0)+x_2^{k+1}(0)y_1'(0)-y_2^{k+1}(0)x_1'(0)-x_1^{k+1}(0)y_2'(0))}{y_1^{k+2}(0)-y_2^{k+2}(0)-mx_1^{k+2}(0)+mx_2^{k+2}(0)}$$

$$\lim_{t\to 0} Y(k)=$$

$$m(\frac{(k+2)(y_1^{k+1}(0)x_2'(0)+x_2^{k+1}(0)y_1'(0)-y_2^{k+1}(0)x_1'(0)-x_1^{k+1}(0)y_2'(0))}{y_1^{k+2}(0)-y_2^{k+2}(0)-mx_1^{k+2}(0)+mx_2^{k+2}(0)})$$

These equations are equal to X(k+1) and Y(k+1):

X(k+1)=

$$\frac{(k+2)(y_1^{k+1}(0)x_2'(0)+x_2^{k+1}(0)y_1'(0)-y_2^{k+1}(0)x_1'(0)-x_1^{k+1}(0)y_2'(0))}{y_1^{k+2}(0)-y_2^{k+2}(0)-mx_1^{k+2}(0)+mx_2^{k+2}(0)}$$

Y(k+1)=

$$m(\frac{(k+2)(y_1^{k+1}(0)x_2'(0)+x_2^{k+1}(0)y_1'(0)-y_2^{k+1}(0)x_1'(0)-x_1^{k+1}(0)y_2'(0))}{y_1^{k+2}(0)-y_2^{k+2}(0)-mx_1^{k+2}(0)+mx_2^{k+2}(0)})$$

Thus, by induction we have shown that (X(n), Y(n)) are the coordinates of the intersection of the line AB and its limit line. Also, notice that the last 4 terms of the numerator will always equal 0 and the first derivatives of the x- and y-parameters are equal for any n>2, so the equations can be simplified to:

$$X(n) = \frac{(n+1)(y_1{}^n(0)x_1'(0) + x_2{}^n(0)y_1'(0) - y_2{}^n(0)x_1'(0) - x_1{}^n(0)y_1'(0))}{y_1{}^{n+1}(0) - y_2{}^{n+1}(0) - mx_1{}^{n+1}(0) + mx_2{}^{n+1}(0)}$$

$$Y(n) = m(\frac{(n+1)(y_1{}^n(0)x_1'(0) + x_2{}^n(0)y_1'(0) - y_2{}^n(0)x_1'(0) - x_1{}^n(0)y_1'(0))}{y_1{}^{n+1}(0) - y_2{}^{n+1}(0) - mx_1{}^{n+1}(0) + mx_2{}^{n+1}(0)})$$

[The case of n=1 is a special condition because the first derivatives are not equal. The equation of the coordinates is:

$$(\frac{2(y_1'(0)x_2'(0) - y_2'(0)x_1'(0))}{y_1''(0) - y_2''(0) - mx_1''(0) + mx_2''(0)}, m(\frac{2(y_1'(0)x_2'(0) - y_2'(0)x_1'(0))}{y_1''(0) - y_2''(0) - mx_1''(0) + mx_2''(0)}))$$]

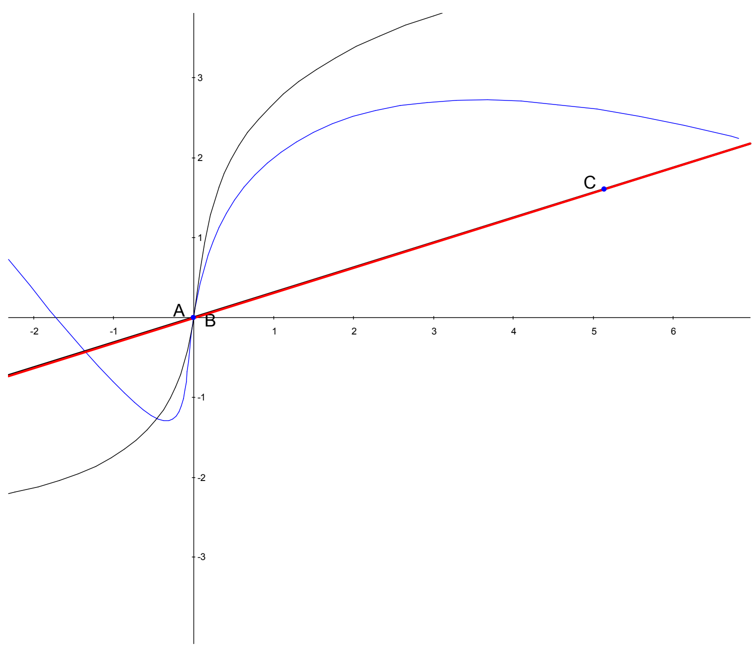

Figure 3: Points A and B move towards the origin according to two different functions. When they coalesce, the limit of the line AB (in red) is equal to the line in black, and the intersection of the two lines (point C) is still