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Special Relativity with Geometry Expressions

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Abstract: Some aspects of special relativity are illustrated using the combination of Geometry Expressions and Mathematica.

1. Time Dilation and Lorentz Contraction

Consider figure 1, a snapshot from a Geometry-Expressions (GX) file. The blue coordinate-frame lines represent our inertial frame, at rest with respect to us. The red coordinate-frame lines represent another inertial frame, one moving at speed v with respect to us. The red x-axis has slope v (actually v / c^2, but c = 1 here) and the red y-axis has slope 1/v, reflecting the space-time symmetry of the Special Theory of Relativity.

B is an event in space-time, that is, a single, fixed point in space and time whose coordinates depend on the frame of reference in which the coordinates are measured. The cyan lines, parallel to the blue axes, measure the coordinates in our frame. The magenta lines, parallel to the red axes, measure the coordinates in the moving frame. Geometry Expressions calculates the Lorentz transformation for us. Minkowski geometry for free.

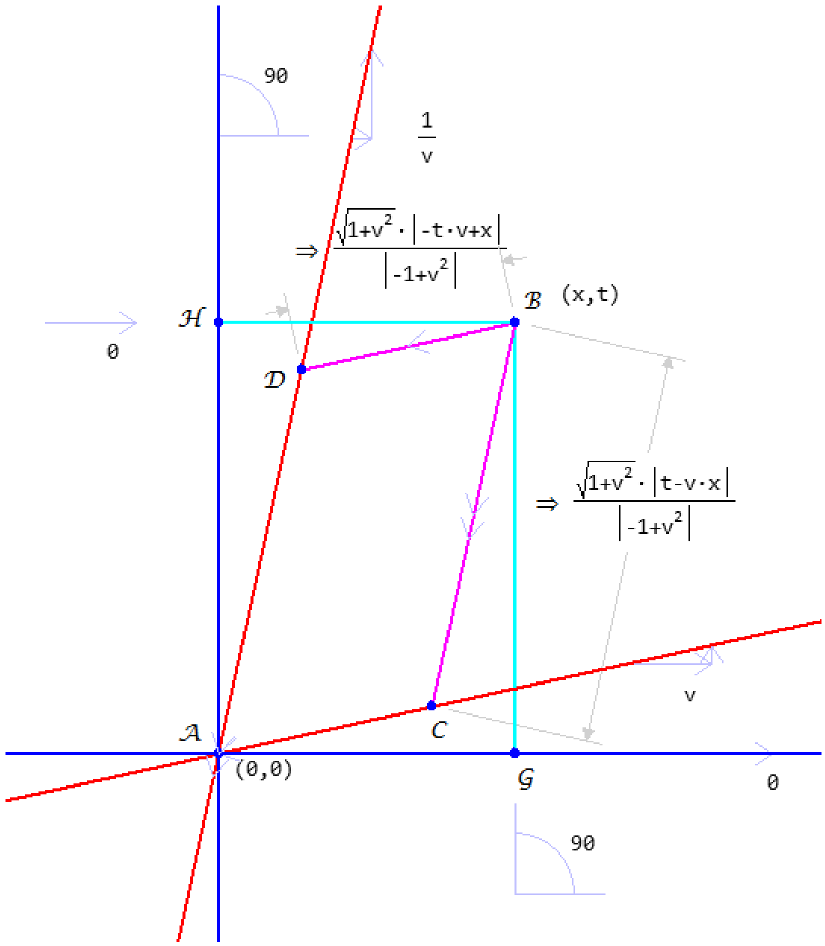


Figure 1: Lorentz contraction (normalized with speed of light = 1)

2. More Minkowski for Free

Here's a nice relativistic analysis of an experiment, building on the framework in the previous section, calculations courtesy of Geometry Expressions once again. Think of laser ranging to the moon. The astronauts left mirrors on the moon, and every day pulses are shot at the mirrors, which are actually corner reflectors, so the pulses bounce straight back. The moon is about 1.5 light-seconds away, so the round trips take about 3 seconds. The moon is moving about 2300 miles per hour, which is about 3e-6 times the speed of light, so the pulses are lengthened by (at most) microseconds, which is very easy to measure with modern equipment, so relativity is spectacularly verified every day in experiments like this.

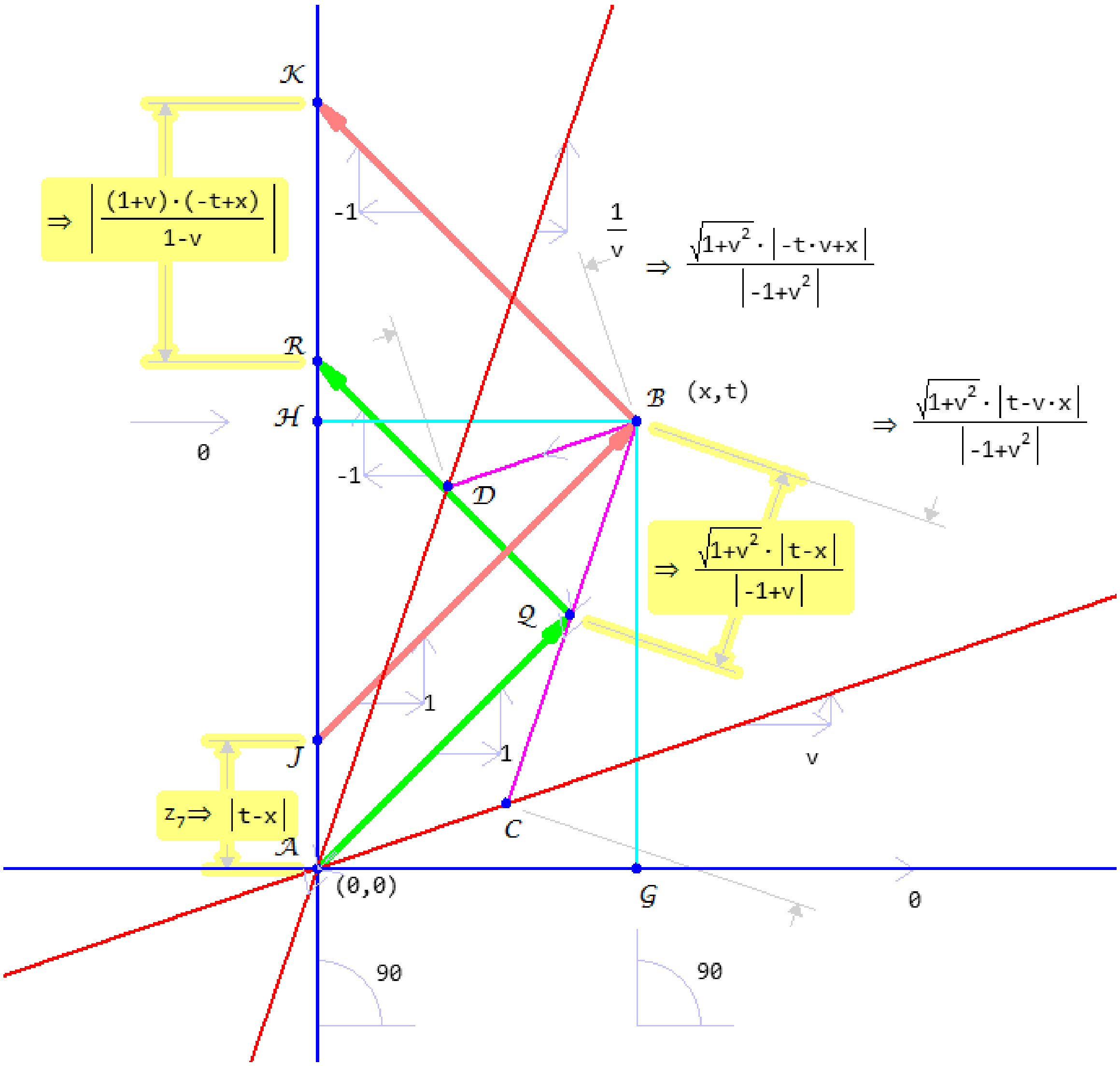


Figure 2: Relativistic Doppler effect.

As before, the blue axes are our non-moving lab frame, and the red axes are a frame of reference moving at velocity v in the x direction. The pulse begins at event A and ends at event J in our blue frame. The red (moon) frame sees the pulse beginning at event Q and ending at event B. The line CB is a line of constant position -- say, the position of the mirror -- in the red moon frame. When we get the pulse back on Earth, it has been stretch to duration RK. This is relativistic Doppler shifting of light.

Figure 3 is a slightly prettier picture.

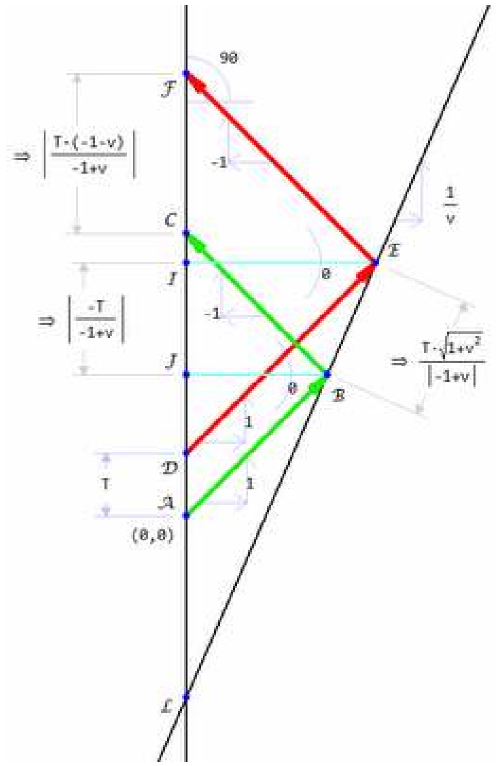


Figure 3: Relativistic doppler effect

3 Double Doppler

The last post showed the relativistic Doppler effect measured in a stationary frame from bouncing signals off an object in a moving frame. How about two moving frames? Geometry Expressions does a really good job, but, this time, the results are sufficiently hairy that we must help out a bit. The Minkowski sketch is simple enough -- one frame has speed u and the other has speed v.

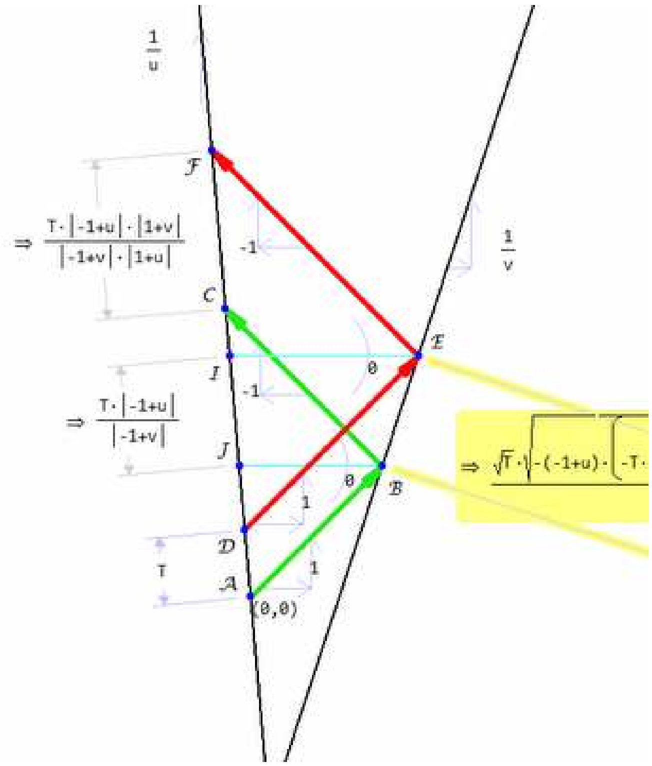
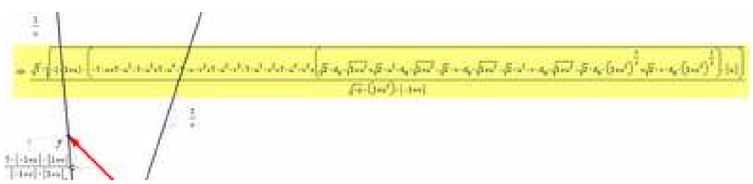


Figure 4: Doppler effect with source and target moving

I chopped off this very hairy expression for the pulse duration in the v frame:



We all know that simplifying algebraic expressions is a tough problem, and sometimes doesn't give the results we want. So, I used GX's "copy to Mathematica form", and, after a FullSimplify in MMA, got this:

Out[68]= $$\frac{\sqrt{T}\,\sqrt{T\,(u-1)^2\,u\,(u^2+1)\,(v^2+1)}}{\sqrt{u}\,(u^2+1)\,|v-1|}$$

This still wasn't good enough for me, so, by hand, I figured this:

Out[70]= $$\frac{T\,(u-1)\sqrt{\frac{v^2+1}{u^2+1}}}{v-1}$$

4. Velocity Addition

Ever go through the traditional derivation of the relativistic formula for addition of velocities? Painful, because there are a bunch of things you have to hold in your head for the duration of a calculation. Modern tools make this easy, almost obvious. First, recall figure 1, in which Geometry Expressions calculates the Lorentz transformation. Now, suppose the point B really represents the endpoint of a particle path, the beginning-point being the origin (0,0). In the blue frame -- our lab frame -- the (average) velocity of the particle over this path is x/t, which we can write as u. What's the velocity of the same particle path, this time measured in the red frame, moving with velocity v w.r.t. our blue frame? Why, let's let Mathematica compute it for us. Use the "Edit / Copy As / Mathematica" menu item in Geometry Expressions on the output expressions in the sketch above, and just paste into Mathematica:

```
In[97]:= xprime =
          (Abs[((v*t*(-1)) + x)] * (Abs[((-1) + (v)^(2))])^((-1)) *
            ((1 + (v)^(2)))^(1/2))
```

Out[97]= $\dfrac{\sqrt{v^2+1}\,|x - t\,v|}{|v^2-1|}$

```
In[98]:= tprime =
          (Abs[(t + (x*v*(-1)))] * (Abs[((-1) + (v)^(2))])^((-1)) *
            ((1 + (v)^(2)))^(1/2))
```

Out[98]= $\dfrac{\sqrt{v^2+1}\,|t - v\,x|}{|v^2-1|}$

```
In[99]:= xprime / tprime
```

Out[99]= $\dfrac{|x - t\,v|}{|t - v\,x|}$

```
In[100]:= (x - t v) / (t - v x)
```

Out[100]= $\dfrac{x - t\,v}{t - v\,x}$

```
In[101]:= % /. (x -> u t) // Simplify
```

Out[101]= $\dfrac{v - u}{u\,v - 1}$

Beautiful simplicity itself. Once again, I've been lazy with absolute-value expressions, just sidestepping them. This means the signs -- but only the signs -- may turn out wrong and we would have to use common sense to correct them.