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Triangle Circle Limits

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Abstract: We examine the limit behavior of various triangle circles, called "limit circles", as the vertices of the triangle coalesce.

Introduction

To aid my research I used mathematical software programs, most importantly Geometry Expressions and Maple. The use of technology allowed me to complete this research topic successfully. I was able to visualize the problems I was trying to solve and eliminate a great deal of error from my work by utilizing these programs. This allowed me to use creativity in studying my topic and quickly find solutions to the problems I faced.

Limit Circles

A "limit circle" is an extension of the concept of single dimension interpolation where two or more points have coalesced. The limit circumcircle is a simple, non-trivial case of multidimensional interpolation through points that have coalesced. We will see that the limit circumcircle has some unexpected properties, especially in terms of its relation to the other triangle circles.

Limit Circumradius – Two Coalescing Points:

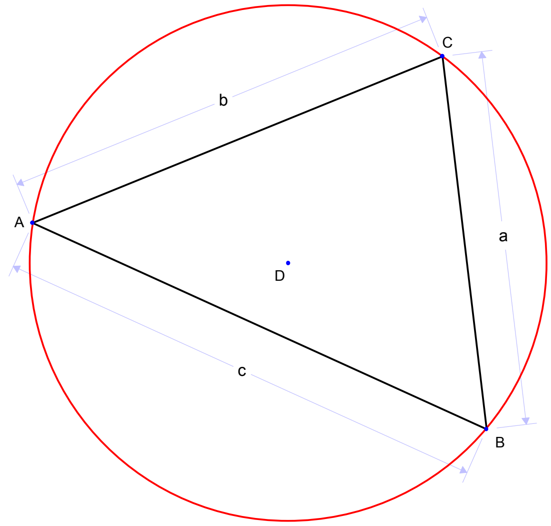

Figure 1. Triangle ABC with circumcircle

We start by examining the limit of the circumradius of a triangle as two of the vertices coalesce. To define the radius of the limit circle, we examine the behavior of the circle as vertices B and C approach one another.

Using the Law of Sines, we know that the circumradius R of triangle ABC with sides a, b, and c opposite the angles A, B, and C is $R = \dfrac{a}{2\sin(A)}$. When vertices B and C coalesce, side a and angle A collapse. Thus we can define side a and angle A as the two functions a(t) and A(t), such that a(0)=0 and A(0)=0. We take the limit of R as t approaches 0:

$$\lim_{t\to 0}\left(\frac{a(t)}{2\sin(A(t))}\right) = 0/0$$

Apply l'Hopital's rule to find the limit radius:

$$\frac{a'(0)}{2\cos(A(0))A'(0)} = \frac{a'(0)}{2A'(0)}$$

But what happens if the derivatives are equal to 0? $\dfrac{a'(0)}{2A'(0)}$ now is the base case where n=1 of the function $R(n) = \dfrac{a^n(0)}{2A^n(0)}$ where $a'(0) = a''(0) = \ldots = a^{n-1}(0) = 0$ and $A'(0) = A''(0) = \ldots = A^{n-1}(0) = 0$. R(1) holds, so we will now prove R(n) for all n>1. Assume R(k) is true. In the next case k+1, the derivatives of a(t) and A(t) equal 0 up until k derivations:

$$a'(0) = a''(0) = ... = a^k(0) = 0$$

$$A'(0) = A''(0) = ... = A^k(0) = 0$$

R(k) now equals 0/0, so we apply l'Hopital's rule, which gives us $\frac{a^{k+1}(0)}{2A^{k+1}(0)}$, which is equal to R(k+1). Thus, by induction we have shown that R(n) is the radius of the limit circumcircle as two of the vertices of the triangle coalesce.

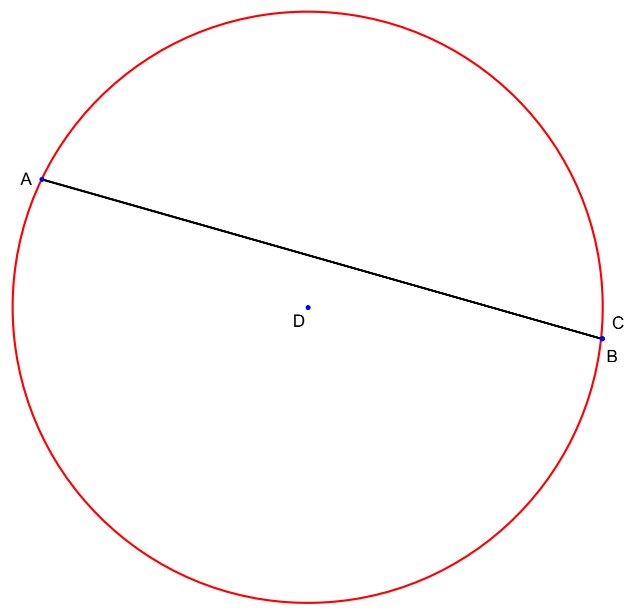

Figure 2. Triangle ABC has collapsed into a line, but the limit circumcircle exists

Limit Circumradius – Three Coalescing Points

Next we examine the limit of the circumcircle of a triangle as all three of its vertices coalesce. Once again, we use the Law of Sines to define the circumradius:

$$R = \frac{a}{2\sin(A)}$$

In any situation where the three vertices coalesce, sides a, b, and c approach 0. However, angles A, B, and C can approach any value. We once again define side a and angle A in terms of the functions a(t) and A(t) such that a(0)=0 and try to find the limit circumradius by taking the limit of R as t approaches 0:

$$\lim_{t\to 0}\left(\frac{a(t)}{2\sin(A(t))}\right)$$

Clearly the circumradius will be 0 unless sin(A(0))=0 so we can apply l'Hopital's rule. Therefore, A(0) must either equal 0 or 180 degrees. We now have the same situation as two coalescing points. Following the same steps as last time, we find the circumradius as three points coalesce is either $\frac{a^n(0)}{2A^n(0)}$ when A(0)=0 or $-\frac{a^n(0)}{2A^n(0)}$ when A(0)=180, where the derivatives of a(t) and A(t) are 0 up to n-1 derivations.

Limit Circle Location

Now that we have defined the radius of the limit circumcircle, we will examine where this circle is located with respect to the three points that define it. We start with two coalescing vertices. If we move two points towards each other along any curve, the line formed by these two points is an approximation of the slope of the curve. The approximation becomes more and more accurate until the two points coalesce, when the slope of the line at the limit becomes the instantaneous slope of the curve where the two points coalesced. The line is now tangent to the curve.

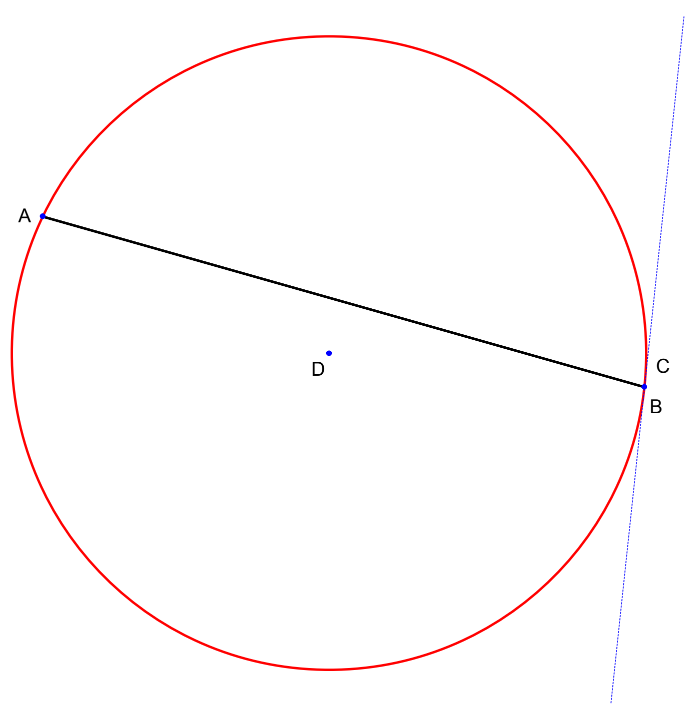

Figure 3. Limit circumcircle is tangent to the path along which points coalesce

It seems that it is possible to apply this to a limit circumcircle. The circle formed by two coalescing points should be tangent to the line formed by the coalescing vertices at the point at which they coalesce.

To prove this, we define the points separately, rather than as vertices of a triangle. We then examine two circles: one that intersects the three points, and one that intersects one point and is tangent to the line formed by the other two at one of the points. If we can show that at the limit the two circles are equivalent, then we have shown that a limit circumcircle is tangent to the line formed by two coalescing points.

Let points A and C be defined parametrically and point B be set to coordinates (1,0).

Point A: $X=x_1(t)$, $Y=y_1(t)$

Point C: $X=x_2(t)$, $Y=y_2(t)$

Such that $x_1(0) = y_1(0) = x_2(0) = y_2(0) = 0$.

Equation of a circle formed by points A, B, and C:

F:=

$$X^2 + Y^2 + X\frac{(-y_2(t) - x_2(t)^2 y_1(t) - y_2(t)^2 y_1(t) + y_2(t)y_1(t)^2 + y_2(t)x_1(t)^2 + y_1(t))}{x_2(t)y_1(t) - y_1(t) - y_2(t)x_1(t) + y_2(t)} +$$

$$Y\frac{(y_2(t)^2 x_1(t) + x_2(t)^2 x_1(t) - x_2(t)y_1(t)^2 - x_1(t) - x_2(t)x_1(t)^2 - y_2(t)^2 + y_1(t)^2 + x_2(t) + x_1(t)^2 - x_2(t)^2)}{x_2(t)y_1(t) - y_1(t) - y_2(t)x_1(t) + y_2(t)} +$$

$$\frac{x_2(t)y_1(t) + y_2(t)x_1(t) - y_2(t)x_1(t)^2 - y_2(t)y_1(t)^2 + y_2(t)^2 y_1(t) + x_2(t)^2 y_1(t)}{x_2(t)y_1(t) - y_1(t) - y_2(t)x_1(t) + y_2(t)} = 0$$

Limit of F as t approaches 0:

$$X^2 + \frac{X(y_1'(0) - y_2'(0))}{y_2'(0) - y_1'(0)} + Y^2 + \frac{Y(x_2'(0) - x_1'(0))}{y_2'(0) - y_1'(0)} = 0$$

Equation of a circle that intersects point B and is tangent to the line AC at point A:

G:=

$$X^2+Y^2+X\frac{y_1(t)+y_1(t)x_1(t)^2-2y_1(t)x_2(t)x_1(t)+y_1(t)^3-y_2(t)+y_2(t)x_1(t)^2-y_2(t)y_1(t)^2}{-y_1(t)+y_1(t)x_2(t)+y_2(t)-y_2(t)x_1(t)}+$$

$$Y\frac{-x_1(t)+2x_1(t)^2-x_1(t)^3+x_2(t)-2x_2(t)x_1(t)+x_2(t)x_1(t)^2+2y_1(t)^2-y_1(t)^2x_1(t)-y_1(t)^2x_2(t)-2y_2(t)y_1(t)+2y_2(t)y_1(t)x_1(t)}{-y_1(t)+y_1(t)x_2(t)+y_2(t)-y_2(t)x_1(t)}$$

$$+\frac{y_2(t)x_1(t)-y_1(t)x_2(t)+y_2(t)y_1(t)^2-y_1(t)x_1(t)^2-y_2(t)x_1(t)^2-y_1(t)^3+2y_1(t)x_2(t)x_1(t)}{-y_1(t)+y_1(t)x_2(t)+y_2(t)-y_2(t)x_1(t)}=0$$

Limit of G as t approaches 0:

$$X^2+\frac{X(y_1'(0)-y_2'(0))}{y_2'(0)-y_1'(0)}+Y^2+\frac{Y(x_2'(0)-x_1'(0))}{y_2'(0)-y_1'(0)}=0$$

We can extend this to the limit circumcircle formed by three coalescing vertices. Because one of the angles must equal 180 degrees at the limit for the limit circle to exist, the three lines formed by the vertices must be equal at the limit. As we have shown, the limit circle is tangent to the line formed by two coalescing points at the limit. Since the three lines are equivalent, the limit circumcircle is tangent to all three lines at the point at which the three vertices coalesce.

An Example:

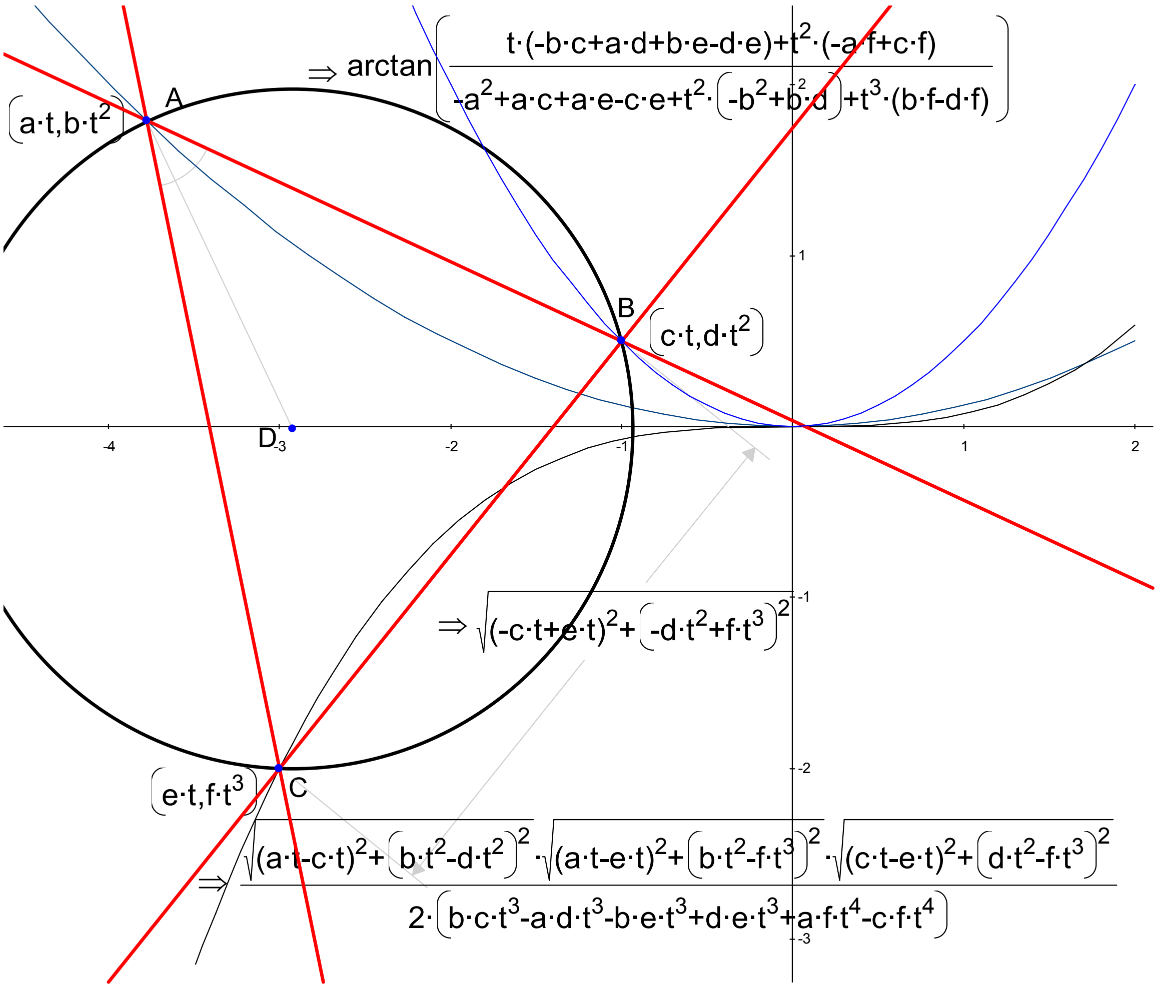

Figure 4. A specific merge path for coalescing points

Points A, B, and C move along different parametric functions such that lines AB, BC, and CA are equal when the points coalesce at the origin. Using Geometry Expressions we calculate the symbolic radius of the circle. Then using Maple, we take the limit as t approaches 0 to find the limit radius:

```
> limit(1/2*((t*c-t*e)^2+(t^2*d-t^3*f)^2)^(1/2)*((t*a-
t*e)^2+(t^2*b-t^3*f)^2)^(1/2)*((t*a-t*c)^2+(t^2*b-
t^2*d)^2)^(1/2)/(t^3*c*b-t^3*d*a-t^3*e*b+t^3*e*d+t^4*f*a-
t^4*f*c), t=0, left);
```

$$\frac{1}{2}\,\frac{\sqrt{(c-e)^2}\,\sqrt{(a-e)^2}\,\sqrt{(a-c)^2}}{-c\,b+d\,a+e\,b-e\,d}$$

```
> simplify(1/2*((c-e)^2)^(1/2)*((a-e)^2)^(1/2)*((a-
c)^2)^(1/2)/(-c*b+d*a+e*b-e*d),'symbolic');
```

$$\frac{1}{2}\frac{(c-e)(a-e)(a-c)}{-c\,b+d\,a+e\,b-e\,d}$$

Then we calculate angle A and the opposite side BC. We put these two equations into Maple in the form (side)/(2(angle)) and take the limit as t approaches 0:

```
> limit(1/2*((-t*c+t*e)^2+(-t^2*d+t^3*f)^2)^(1/2)/arctan(((-c*b+d*a+e*b-e*d)*t+(-f*a+f*c)*t^2)/(-a^2+c*a+e*a-e*c+(-b^2+d*b)*t^2+(f*b-f*d)*t^3)), t=0, left);
```

$$\frac{1}{2}\frac{\sqrt{(c-e)^2}\,(a^2-c\,a-e\,a+e\,c)}{-c\,b+d\,a+e\,b-e\,d}$$

```
> factor(1/2*((c-e)^2)^(1/2)/(-c*b+d*a+e*b-e*d)*(a^2-c*a-e*a+e*c));
```

$$\frac{1}{2}\frac{\sqrt{(c-e)^2}\,(a-e)(a-c)}{-c\,b+d\,a+e\,b-e\,d}$$

```
> simplify(1/2*((c-e)^2)^(1/2)*(a-e)*(a-c)/(-c*b+d*a+e*b-e*d),'symbolic');
```

$$\frac{1}{2}\frac{(c-e)(a-e)(a-c)}{-c\,b+d\,a+e\,b-e\,d}$$

The two equations are equal.

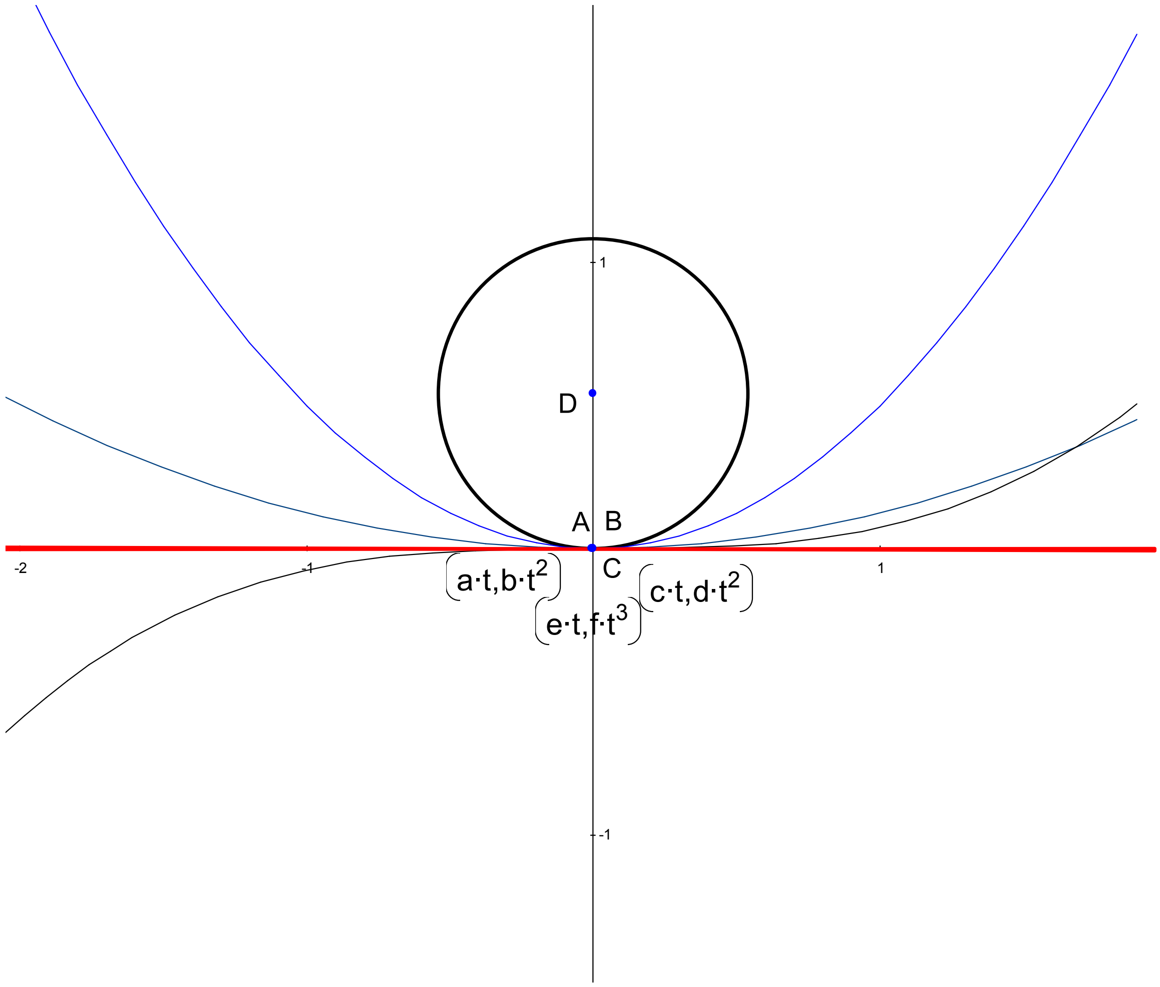

Figure 5. Limit circumcircle for the example of figure 4

Various Limit Triangle Circles – Three Coalescing Vertices:

Interestingly, many of the various circles defined by a triangle can be defined very simply in terms of the circumradius when the three vertices of the triangle coalesce.

We know that at the limit when the three vertices of the triangle ABC coalesce, angle A approaches 180 degrees, angle B approaches 0 degrees, and angle C approaches 0 degrees while sides a, b, and c approach 0. One important property of the limit triangle is that the Brocard angle ω approaches 0. If we take the limit of the equation of the Brocard angle:

$$\omega = \arctan\left(\frac{\sin A \sin B \sin C}{1 + \cos A \cos B \cos C}\right)$$

We find that ω =0. Another property is the fact that the rate of change of the semiperimeter, s, is equal to the rate of change of side a at the limit. Using the Law of Sines we know that $b' = \frac{-B'a'}{A'}$ and $c' = \frac{a'(A'+B')}{A'}$. If we take the derivative of s, we get $s' = \frac{a'+b'+c'}{2}$. Substituting these values, we get a'.

Using these properties of a limit triangle, we can find the radii of various other triangle circles. I will use the following format: lim(Original Radius)=Limit Radius. Note that R is the circumradius, and r is the inradius. As will be shown later, r=0 at the limit.

Anticomplentary Circle:

$$\lim(2R) = 2R$$

Apollonius Circle:

The radius of the Circle of Apollonius is not directly related to the circumradius.

Bevan Circle:

$$\lim(2R) = 2R$$

Brocard Circle:

$$\lim\left(\frac{R\sqrt{1-4\sin^2(\omega)}}{2\cos(\omega)}\right) = R/2$$

Conway Circle:

$$\lim\left(\sqrt{r^2+s^2}\right) = 0$$

Cosine Circle:

$$\lim(R\tan(\omega)) = 0$$

De Longchamps Circle:

$$\lim\left(4R\sqrt{-\cos A\cos B\cos C}\right) = 4R$$

Excircles:

$$\lim(4R\sin(A/2)\cos(B/2)\cos(C/2)) = 4R$$

Note that this equation only applies to the excircle opposite angle A. The excircles opposite angles B and C have radii of 0 because sin(0)=0

Extangents Circle:

The limit radius appears to be R/2.

Fuhrmann Circle:
The radius of the Fuhrmann Circle is the distance between the incenter and circumcenter of the triangle, determined by the equation $\sqrt{R(R-2r)}$.

$$\lim\left(\sqrt{R(R-2r)}\right)=R$$

Gallatly Circle:

$$\lim\left(R\sin(\omega)\right)=0$$

Half-Altitude Circle:

The limit radius appears to be 2R.

Half-Moses Circle:

$$\lim\left(\frac{R\tan(\omega)\sin(2\omega)}{2}\right)=0$$

Hexyl Circle:

$$\lim\left(2R\right)=2R$$

Incenter-Excenter Circle:

$$\lim\left(2R\sin(A/2)\right)=2R$$

Incentral Circle:

The radius of the Incentral Circle is not directly related to the circumradius at the limit.

Incircle:

$$\lim\left(4R\sin(A/2)\sin(B/2)\sin(C/2)\right)=0$$

Inner Napoleon Circle:

$$\lim\left(\frac{\sqrt{a^2+b^2+c^2-4\sqrt{3}\Delta}}{3\sqrt{2}}\right)=0$$

Δ is the area of the triangle.

Inner Soddy Circle:

Because the Soddy Circles have radii of 0 at the limit, the Inner Soddy Circle which is externally tangent to them will also have a limit radius of 0.

Intangents Circle:

$$\lim\left(\frac{r}{4\left|\cos A\cos B\cos C\right|}\right)=0$$

Johnson-Yff Circles:

$$\lim\left(\frac{rR}{r \pm R}\right) = 0$$

Kenmotu Circle:

$$\lim\left(\frac{\sqrt{2}R\sin\omega}{\cos\omega + \sin\omega}\right) = 0$$

Longuet-Higgins Circle:

$$\lim\left(\sqrt{16R^2 - (a+b+c)^2}\right) = 4R$$

Lucas Circles:

$$\lim\left(\frac{R}{1 + \frac{2aR}{bc}}\right) = 0$$

Lucas Circles Radical Circle:

$$\lim\left(\frac{R}{\cot\omega + 2}\right) = 0$$

Lucas Inner Circle:

$$\lim\left(\frac{R}{\cot\omega + 7}\right) = 0$$

Malfatti Circles:

The Malfatti Circles are in the interior of the triangle, so at the limit they collapse to a point.

Mandart Circle:

$$\lim\left(\frac{s\sqrt{(4R^2 - bc)(4R^2 - ca)(4R^2 - ab)}}{abc}\right) = \infty$$

Mixtilinear Circle:

The radius of the Mixtilinear Circle is not directly related to the circumradius at the limit.

Mixtilinear Incircle:

The radius of the Mixtlinear Incircle is not directly related to the circumradius at the limit.

Moses Circle:

$$\lim\left(R\tan(\omega)\sin(2\omega)\right)=0$$

Nine-Point Circle:

$$\lim\left(R/2\right)=R/2$$

Orthoptic Circle of the Steiner Inellipse:

$$\lim\left(\frac{\sqrt{a^2+b^2+c^2}}{\sqrt{3}}\right)=0$$

Outer Napoleon Circle:

$$\lim\left(\frac{\sqrt{a^2+b^2+c^2+4\sqrt{3}\Delta}}{3\sqrt{2}}\right)=0$$

Δ is the area of the triangle.

Pedal Circles:

The radius of a Pedal Circle is not directly related to the circumradius at the limit.

Polar Circle:

$$\lim\left(\sqrt{4R^2-\frac{a^2+b^2+c^2}{2}}\right)=2R$$

Power Circles:
The radii of the power circles at the limit is 0 because the midpoints and vertices of a triangle collapse into one point, and the power circles are centered at the midpoints and intersect the opposite vertex.

Sine-Triple-Angle Circle:

$$\lim\left(\frac{R}{\left|1+8\cos A\cos B\cos C\right|}\right)=R/7$$

Spieker Circle:

$$\lim\left(r/2\right)=0$$

Stammler Circle:

$$\lim\left(2R\right)=2R$$

Stammler Circles:

$$\lim\left(R\sqrt{1+8\cos\left(\frac{B-C}{3}\right)\cos\left(\frac{B+2C}{3}\right)\cos\left(\frac{2B+C}{3}\right)}\right)=3R$$

Note that this is the equation of the Stammler Circle opposite angle A. The other Stammler Circles have radii of 0 at the limit.

Steiner Circle:

$$\lim\left(\frac{3R}{2}\right)=\frac{3R}{2}$$

Symmedial Circle:
The radius of the Symmedial Circle is not directly related to the circumradius.

Tangential Circle:

$$\lim\left(\frac{R}{4\left|\cos A\cos B\cos C\right|}\right)=R/4$$

Taylor Circle:

$$\lim\left(R\sqrt{\sin^2(A)\sin^2(B)\sin^2(C)+\cos^2(A)\cos^2(B)\cos^2(C)}\right)=R$$

Tucker Circle:

$$\lim\left(R\sqrt{\lambda^2+(1-\lambda)^2\tan^2(\omega)}\right)=\lambda R$$

λ is the parameter of any Tucker Circle.

Van Lamoen Circle:
The limit of the radius of the van Lamoen circle appears to be infinite.

Yff Circles:

The radius of an Yff circle is equal to a Johnson-Yff Circle, so the radius is 0 at the limit.

Yiu Circle:
The limit of the radius of the Yiu Circle appears to be R/9.

Yiu Circles:
The limits of the radii of the Yiu Circles appear to be R/3.

A list of these triangle circles and more can be found at http://mathworld.wolfram.com/topics/TriangleCircles.html.

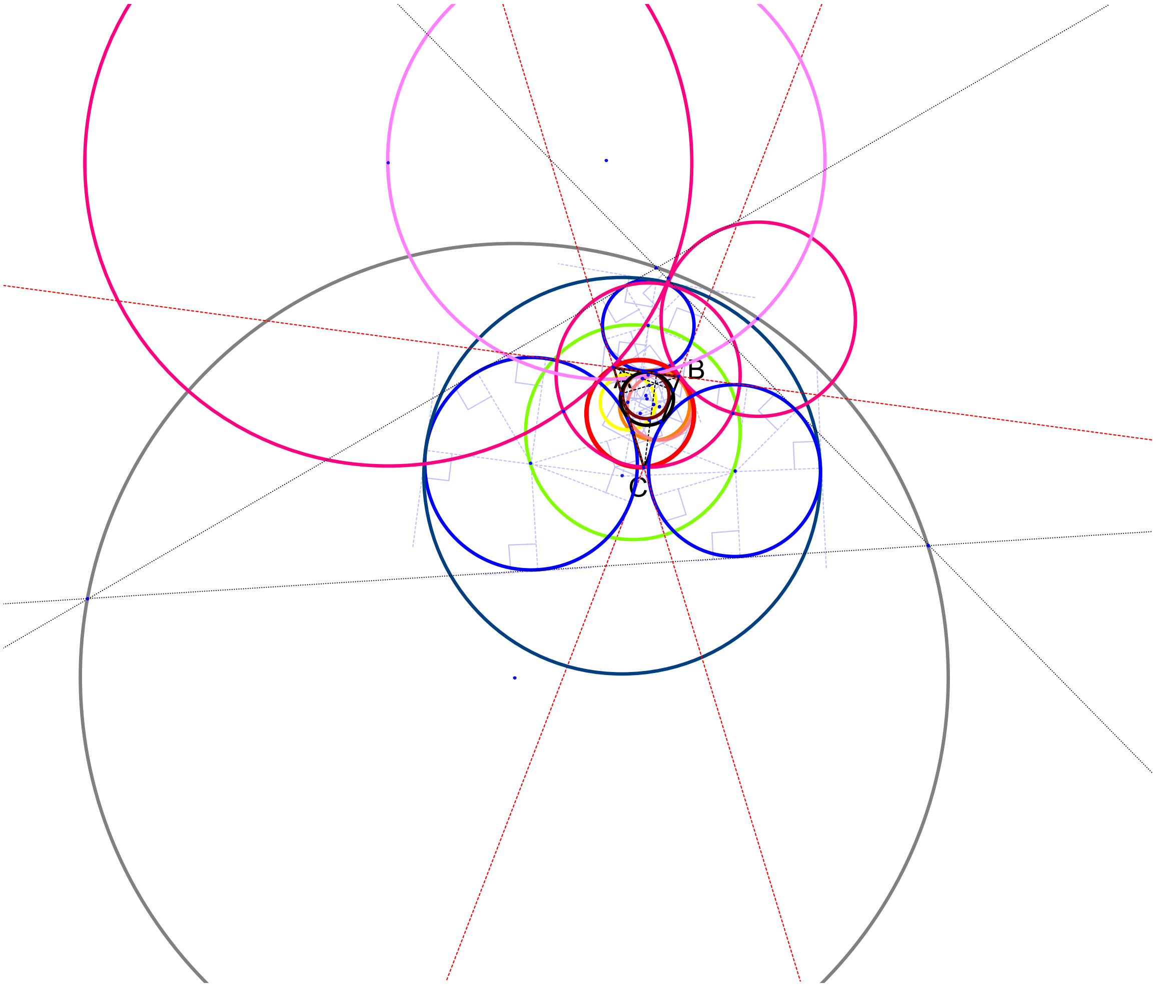

Figure 6. A number of triangle circles defined by triangle ABC. When A, B, and C coalesce, they become much simpler

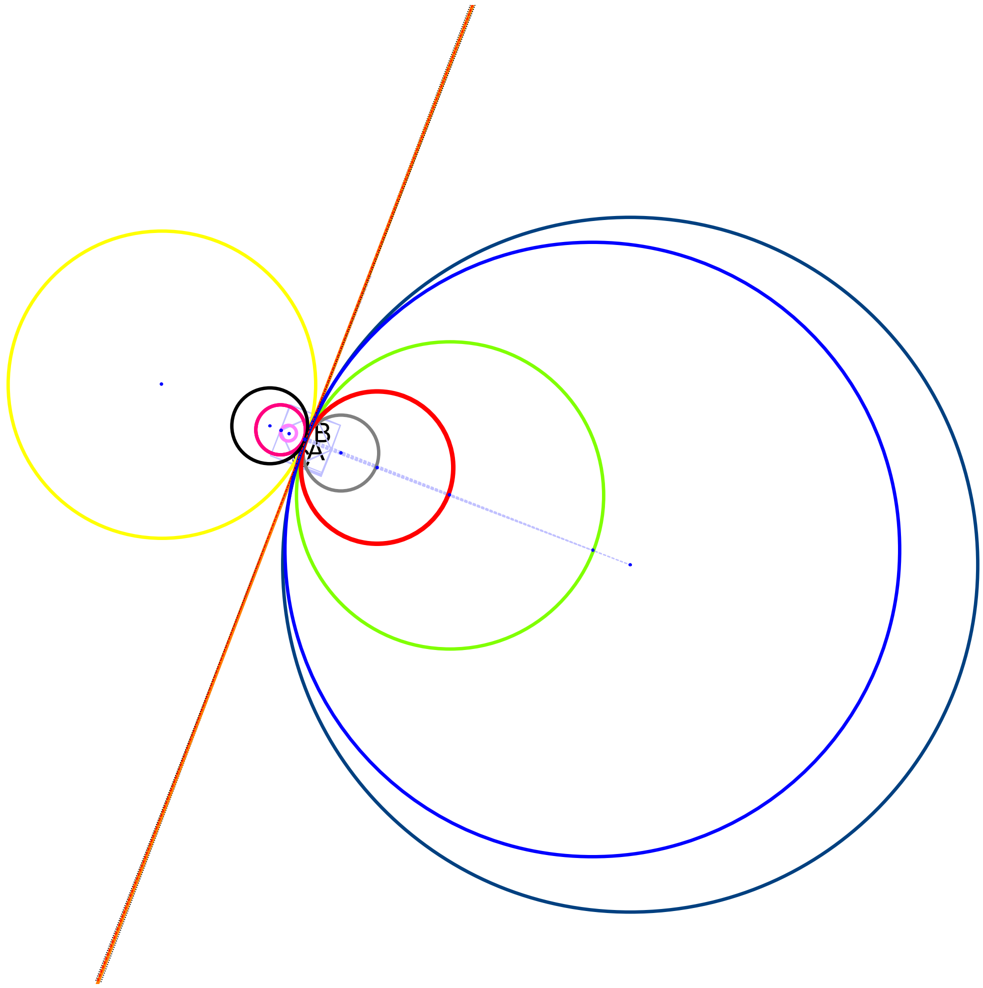

Figure 7. Limits of the cirtcles in figure 6. The circle in this diagram that is not directly defined by the circumradius is the Circle of Apollonius, the largest defined circle in the diagram. Note that one circle in this diagram is not tangent to the lines AB, BC, and CA (the dashed lines): the Yiu Circle, which is also the smallest circle in the diagram. Also note that the Mandart Circle approaches infinity, becoming a line that is equivalent to the lines AB, BC, and CA