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Pedal and Skew Pedal Curves of a Parabola

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Abstract: Properties of the evolute, the pedal curve, and the skew pedal curve are analyzed with Geometry Expressions and Maple. The base curve used is the parabola $y=x^2$

1. Pedal Curve

Given a pole P, the pedal curve is the locus of the intersection of the tangent to the original curve with the perpendicular from P to the tangent. This takes a variety of shapes depending on the location of P.

We derive an equation for the pedal curve with pole (a,b):

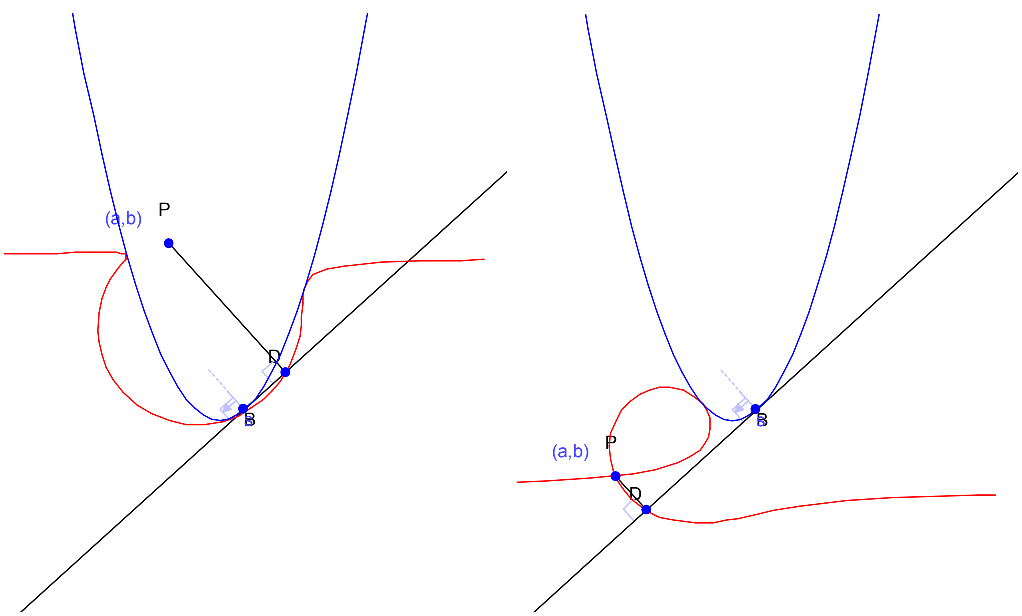


Figure 1. Pedal curve for two different poles

We compute the equation of the pedal curve, both parametric and implicit:

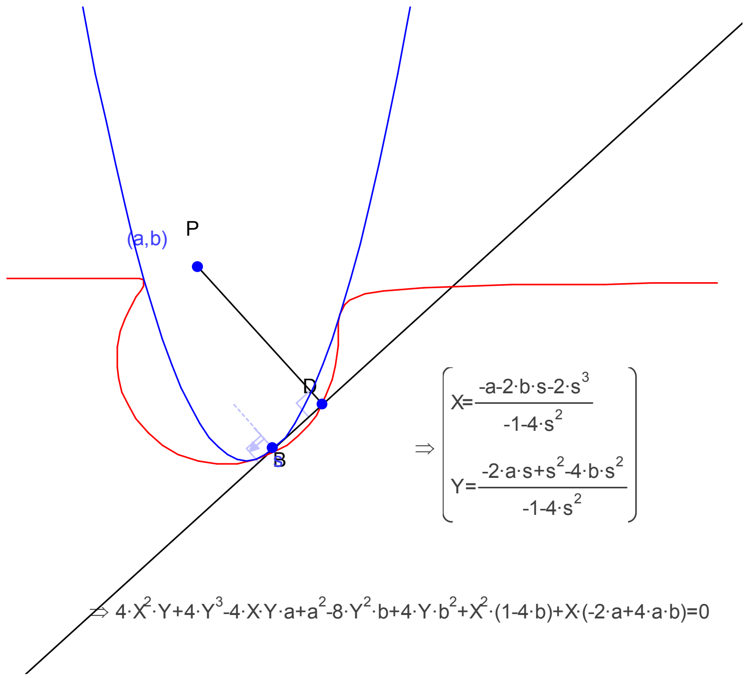


Figure 2: Equation of the pedal curve with pole (a,b)

We can compute the limit of Y as s tends to plus and minus infinity:

```
> limit((-2*s*a+s^2-4*s^2*b)/(-1-4*s^2),s=infinity);
```

$$-\frac{1}{4}+b$$

```
> limit((-2*s*a+s^2-4*s^2*b)/(-1-4*s^2),s=-infinity);
```

$$-\frac{1}{4}+b$$

So we see that the pedal curve is asymptotic to the line

$$y=b-\frac{1}{4}$$

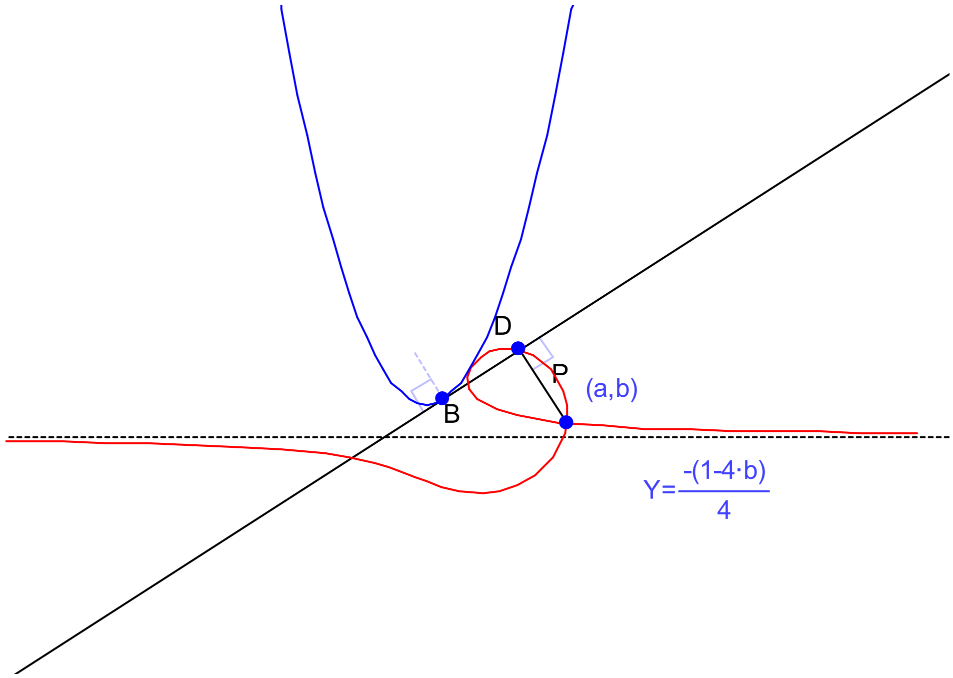


Figure 3: Pedal curve asymptote

Examining Figure 1, we see that, depending on the location of P, the pedal curve can have one or three points of contact with the original parabola. We will attempt to characterize the locations which give rise to these two different circumstances. First, we find the locations of the points of contact. We can do this in Maple by simultaneously solving the implicit equation of the pedal curve and the equation of the parabola ($Y=X^2$)

```
> solve({4*Y*X^2+4*Y^3-4*a*Y*X+a^2-8*b*Y^2+4*b^2*Y+(1-4*b)*X^2+(-2*a+4*b*a)*X = 0,Y=X^2},{X,Y});
```

$$\left\{ Y = \mathrm{RootOf}(2\,\_Z^3 + (1-2\,b)\,\_Z - a)^2, X = \mathrm{RootOf}(2\,\_Z^3 + (1-2\,b)\,\_Z - a) \right\},$$

$$\left\{ Y = \mathrm{RootOf}(2\,\_Z^3 + (1-2\,b)\,\_Z - a)^2, X = \mathrm{RootOf}(2\,\_Z^3 + (1-2\,b)\,\_Z - a) \right\}$$

The X locations for the points of contact are the roots of a particular cubic. We can draw this cubic on the diagram.

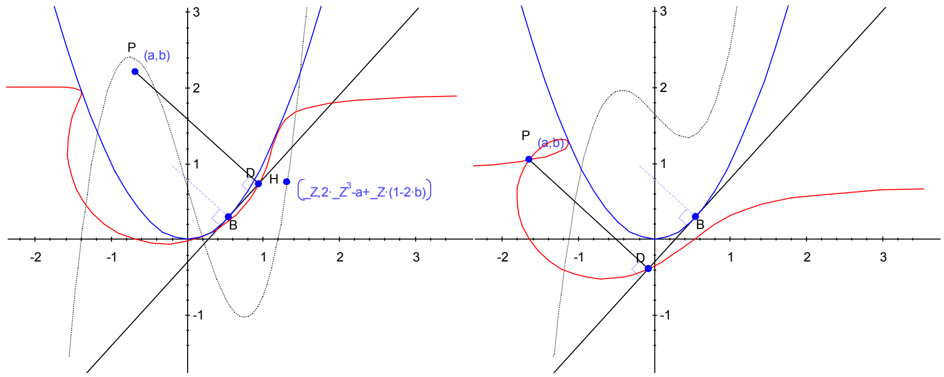


Figure 4: Cubic whose roots give the locations of points of contact. The curve has either 1 or 3 real roots

The dividing line between one and three roots will be when the cubic has a double root. In this case there will be double contact between the pedal curve and the parabola. Two of the roots of the cubic are expressed as a complex conjugate pair.

```
> allvalues(RootOf(2*_Z^3+(1-2*b)*_Z-a));
```

$$\frac{1}{6}\left(54\,a+6\sqrt{6-36\,b+72\,b^2-48\,b^3+81\,a^2}\right)^{(1/3)}-\frac{6\left(\frac{1}{6}-\frac{1}{3}\,b\right)}{\left(54\,a+6\sqrt{6-36\,b+72\,b^2-48\,b^3+81\,a^2}\right)^{(1/3)}},$$

$$-\frac{1}{12}\left(54\,a+6\sqrt{6-36\,b+72\,b^2-48\,b^3+81\,a^2}\right)^{(1/3)}+\frac{3\left(\frac{1}{6}-\frac{1}{3}\,b\right)}{\left(54\,a+6\sqrt{6-36\,b+72\,b^2-48\,b^3+81\,a^2}\right)^{(1/3)}}+$$

$$\frac{1}{2}\,I\sqrt{3}\left(\frac{1}{6}\left(54\,a+6\sqrt{6-36\,b+72\,b^2-48\,b^3+81\,a^2}\right)^{(1/3)}\right.$$

$$\left.+\frac{6\left(\frac{1}{6}-\frac{1}{3}\,b\right)}{\left(54\,a+6\sqrt{6-36\,b+72\,b^2-48\,b^3+81\,a^2}\right)^{(1/3)}}\right),$$

$$-\frac{1}{12}\left(54\,a+6\sqrt{6-36\,b+72\,b^2-48\,b^3+81\,a^2}\right)^{(1/3)}+\frac{3\left(\frac{1}{6}-\frac{1}{3}\,b\right)}{\left(54\,a+6\sqrt{6-36\,b+72\,b^2-48\,b^3+81\,a^2}\right)^{(1/3)}}-\frac{1}{2}$$

$$I\sqrt{3}\left(\frac{1}{6}(54a+6\sqrt{6-36b+72b^2-48b^3+81a^2})^{(1/3)} + \frac{6\left(\frac{1}{6}-\frac{1}{3}b\right)}{(54a+6\sqrt{6-36b+72b^2-48b^3+81a^2})^{(1/3)}}\right)$$

For a double solution, the complex piece has to be 0. We can solve for this in Maple:

```
> solve((1/6*(54*a+6*(6-36*b+72*b^2-48*b^3+81*a^2)^(1/2))^(1/3)+6*(1/6-1/3*b)/(54*a+6*(6-36*b+72*b^2-48*b^3+81*a^2)^(1/2))^(1/3))=0,b);
```

$$\frac{1}{12}54^{(2/3)}a^{(2/3)}+\frac{1}{2}$$

```
> simplify(%);
```

$$\frac{3}{4}2^{(2/3)}a^{(2/3)}+\frac{1}{2}$$

We can feed this value in for b:

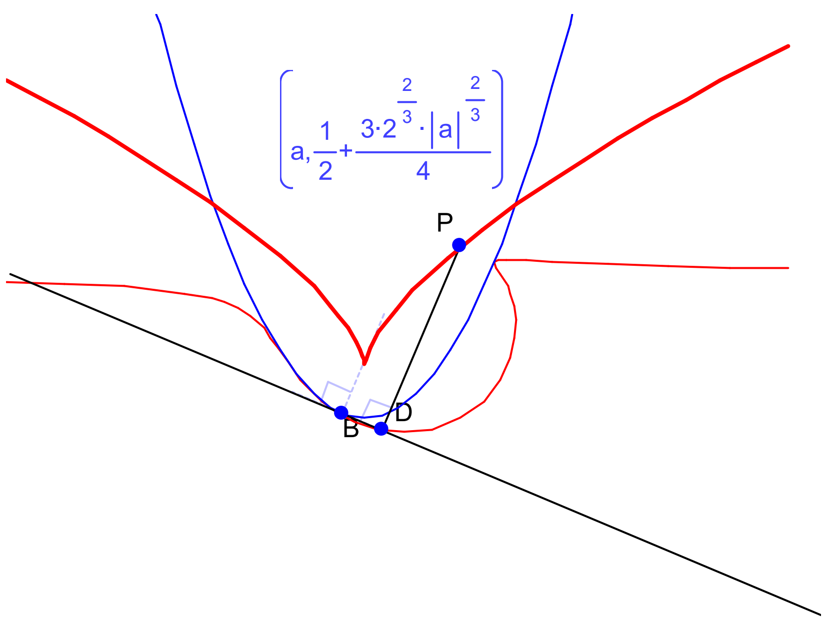


Figure 5: Discriminant curve dividing the region where the pedal curve touches once from the region where the pedal curve touches three times. If P lies on this discriminant curve, the pedal curve has two point contact.

To find the points of contact, we can get the parametric equation of the pedal curve and find values for s where the X coordinate of the pedal curve is equal to s:

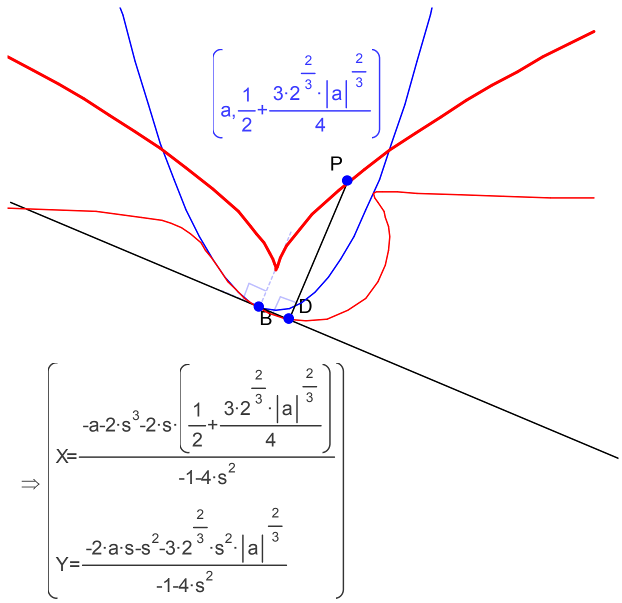


Figure 6: Parametric equation of the pedal curve where the pole lies on the discriminant curve:

```
> solve((-a-2*s^3-2*(3/4*abs(a)^(2/3)*2^(2/3)+1/2)*s)/(-4*s^2-1)=s,s) assuming a>0;
```

$$2^{(1/3)}\, a^{(1/3)},\ -\frac{1}{2}\, 2^{(1/3)}\, a^{(1/3)},\ -\frac{1}{2}\, 2^{(1/3)}\, a^{(1/3)}$$

We can plot these points on the diagram:

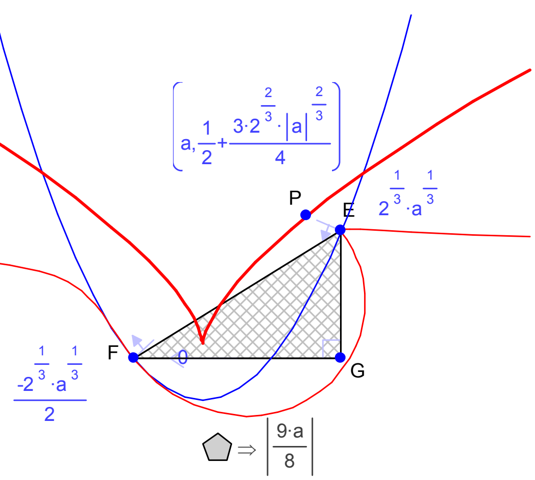


Figure 8: Contact points for critical location of pole. Area of a triangle whose short sides are aligned with the axes and whose hypotenuse joins the points of contact is displayed.

The x coordinate of one point is negative twice the x coordinate of the other. If we create a right triangle whose hypotenuse is the line joining the points of contact, and whose short sides are aligned with the axes, we see that the area of this triangle is simply $\frac{9}{8}a$.

If we create the normal to the parabola (see figure 9), it is easy to see that the normal is everywhere tangent to the discriminant curve.

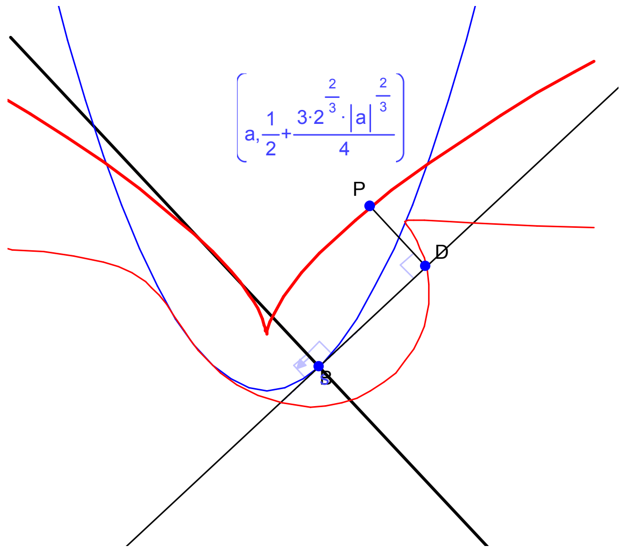


Figure 9: The normal to the parabola is tangent to the discriminant curve.

The discriminant is thus the envelope of the normals to the original curve. The name for this curve is the evolute.

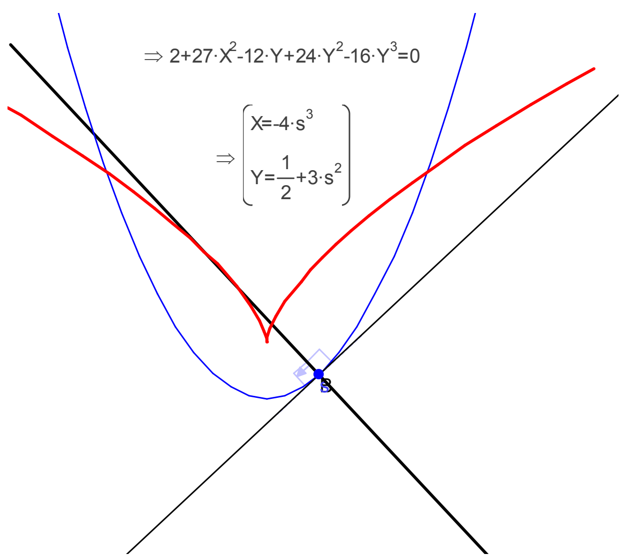


Figure 10: The evolute to the parabola.

It is intuitively reasonable that in general double contact between the pedal curve and the original curve implies a pedal point which lies on the evolute. Proof is left to the reader.

2. Skew Pedal Curves

The pedal curve is defined to be the intersection of the tangent to the original curve with the perpendicular through the pedal point P. We now skew the discussion slightly by examining the locus of the intersection of the tangent to the parabola with the line through P at angle θ to the tangent.

First however, let's examine the skew analog of the evolute: the envelope of the lines making angle θ with the tangent:

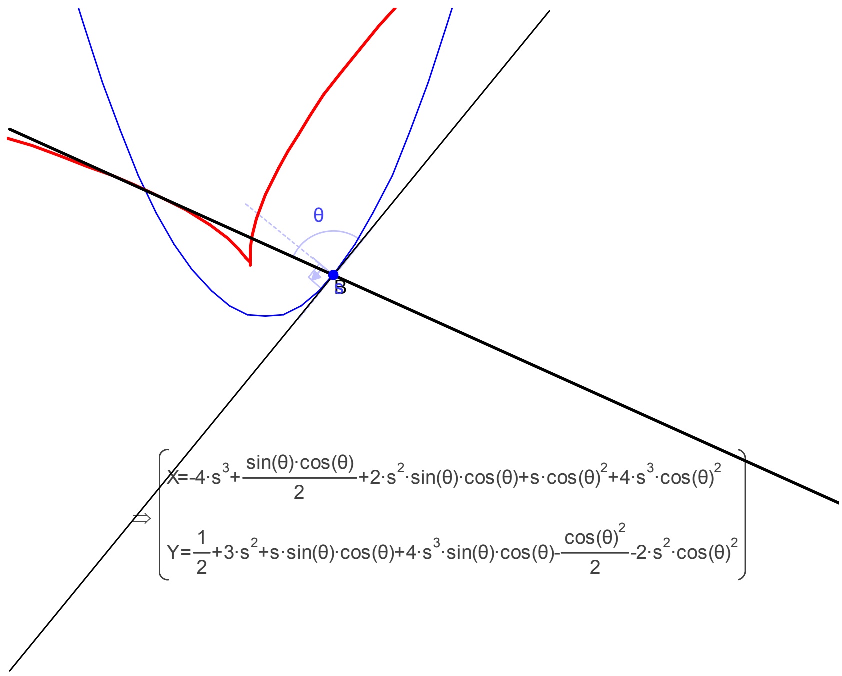


Figure 11: Skew evolute, the envelope of the lines at constant angle to the tangent

We place P at parametric location r on the skew evolute, and construct the skew pedal curve. By construction this is the critical pedal with two point contact.

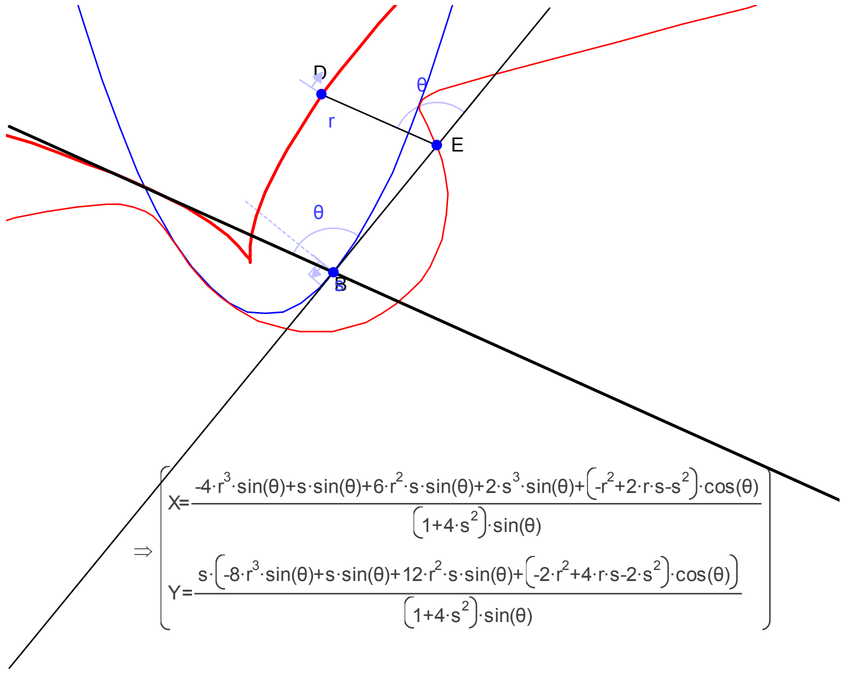


Figure 12: Equation of the critical skew pedal curve

Solving for X=s in Maple gives the parametric location of the intersection points:

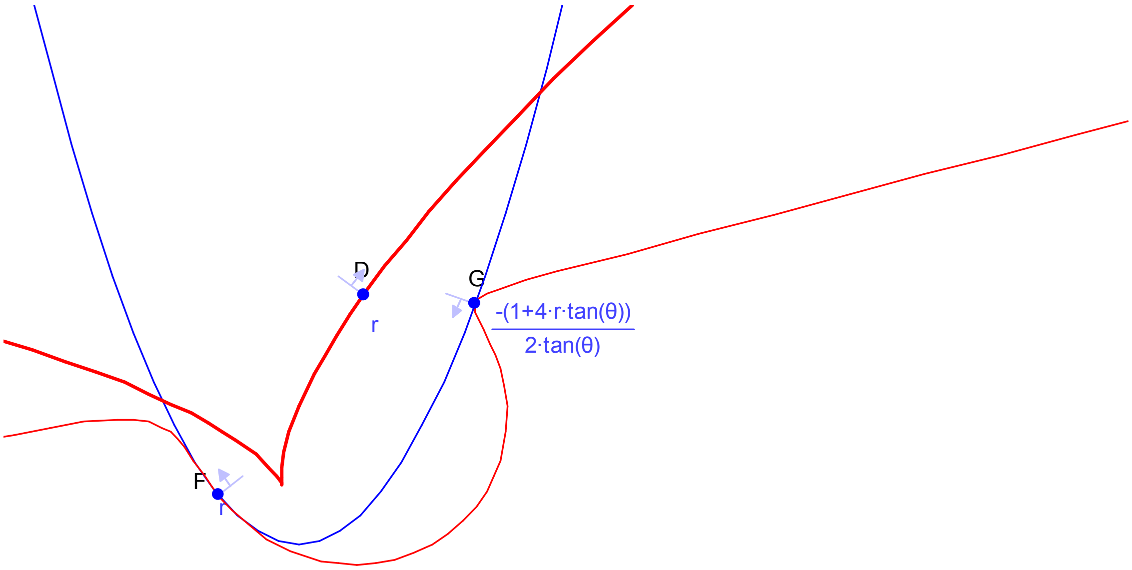
```
> X:=((sin(theta))^((-1))*((sin(theta)*(r)^(3)*(-4))+(sin(theta)*s)+(sin(theta)*s*(r)^(2)*6)+(sin(theta)*(s)^(3)*2)+(cos(theta)*(((r)^(2)*(-1))+(s*r*2)+((s)^(2)*(-1)))))*((1+((s)^(2)*4))^((-1)));
```

$$X := \frac{-4\sin(\theta)\, r^3 + \sin(\theta)\, s + 6\sin(\theta)\, s\, r^2 + 2\sin(\theta)\, s^3 + \cos(\theta)\,(-r^2 + 2\, s\, r - s^2)}{\sin(\theta)\,(4\, s^2 + 1)}$$

```
> solve(X=s,s);
```

$$r,\ r,\ -\frac{1 + 4\tan(\theta)\, r}{2\tan(\theta)}$$

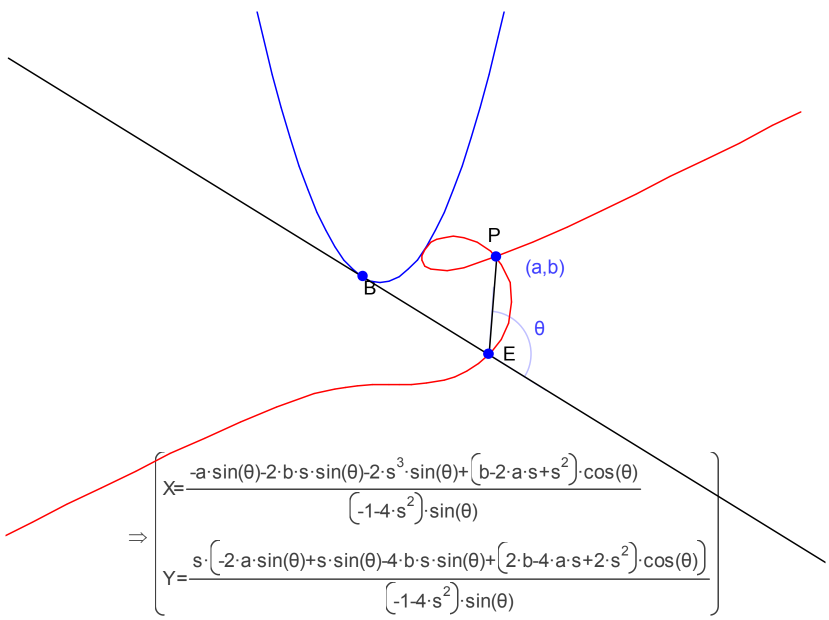
Locating points at these locations on the curve we see they are indeed at the intersections. We can also display the coordinates of P:



$$\Rightarrow \left[-4\cdot r^3+\frac{\sin(\theta)\cdot\cos(\theta)}{2}+2\cdot r^2\cdot\sin(\theta)\cdot\cos(\theta)+r\cdot\cos(\theta)^2+4\cdot r^3\cdot\cos(\theta)^2,\ \frac{1}{2}+3\cdot r^2+r\cdot\sin(\theta)\cdot\cos(\theta)+4\cdot r^3\cdot\sin(\theta)\cdot\cos(\theta)-\frac{\cos(\theta)^2}{2}-2\cdot r^2\cdot\cos(\theta)^2\right]$$

Figure 13: Intersections of the skew pedal with the parabola, along with coordinates of the Pedal point at parametric location r on the skew evolute

Figure 14 shows the equation of the skew pedal curve for general location of the Pedal point:



$$\Rightarrow \left[\begin{array}{l} X=\dfrac{-a\cdot\sin(\theta)-2\cdot b\cdot s\cdot\sin(\theta)-2\cdot s^3\cdot\sin(\theta)+\left(b-2\cdot a\cdot s+s^2\right)\cdot\cos(\theta)}{\left(-1-4\cdot s^2\right)\cdot\sin(\theta)} \\ Y=\dfrac{s\cdot\left(-2\cdot a\cdot\sin(\theta)+s\cdot\sin(\theta)-4\cdot b\cdot s\cdot\sin(\theta)+\left(2\cdot b-4\cdot a\cdot s+2\cdot s^2\right)\cdot\cos(\theta)\right)}{\left(-1-4\cdot s^2\right)\cdot\sin(\theta)} \end{array}\right]$$

Figure 14: Skew Pedal curve with pole (a,b)

If we look at the ratio Y/X, we see this has a limit as X goes to infinity or – infinity:

```
> X := (-sin(theta)*a-2*sin(theta)*s*b-
2*sin(theta)*s^3+cos(theta)*(b-2*s*a+s^2))/(sin(theta)*(-
4*s^2-1));
```

$$X := \frac{-\sin(\theta)\, a - 2\sin(\theta)\, s\, b - 2\sin(\theta)\, s^3 + \cos(\theta)\,(b - 2\, s\, a + s^2)}{\sin(\theta)\,(-4\, s^2 - 1)}$$

```
> Y := (-2*sin(theta)*a+sin(theta)*s-
4*sin(theta)*s*b+cos(theta)*(2*b-
4*s*a+2*s^2))*s/(sin(theta)*(-4*s^2-1));
```

$$Y := \frac{(-2\sin(\theta)\, a + \sin(\theta)\, s - 4\sin(\theta)\, s\, b + \cos(\theta)\,(2\, b - 4\, s\, a + 2\, s^2))\, s}{\sin(\theta)\,(-4\, s^2 - 1)}$$

```
> limit(Y/X,s=infinity);
```

$$-\frac{\cos(\theta)}{\sin(\theta)}$$

```
> limit(Y/X,s=-infinity);
```

$$-\frac{\cos(\theta)}{\sin(\theta)}$$

Let $m = \frac{-1}{\tan(\theta)}$

To get the equation of the limit line we need to simply find

$$C = \lim_{s\to\infty}(Y - mX)$$

Then the asymptote equation is Y=mX+C.

```
> limit(Y+X/tan(theta),s=infinity);
```

$$\frac{4\cos(\theta)\, a\sin(\theta) - 1 + 4\, b - 4\cos(\theta)^2\, b}{4\sin(\theta)^2}$$

Figure 15 shows this line imposed on the curve drawing. We also show the distance of the pedal curve from the asymptote to be the constant

$$\frac{1}{4\sin(\theta)}$$

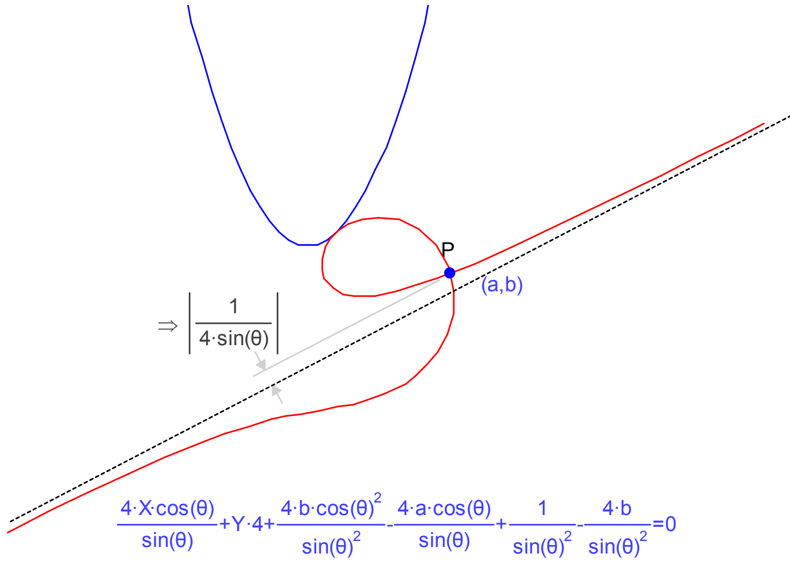


Figure 15: Asymptote line of the skew pedal curve