The Journal of Symbolic Geometry
Volume 1 (2006)

Feynman's and Steiner's Triangle

Philip Todd

Saltire Software

PO Box 1565, Beaverton OR 97075

philt@saltire.com

Abstract: Using a symbolic geometry system, we look at a triangle theorem attributed to Feinman and its more general form due to Steiner.

1. Feinman's Triangle

The triangle shown below is 1/7 the area of the original triangle. This theorem was apparently proved by Feinman over dinner once.

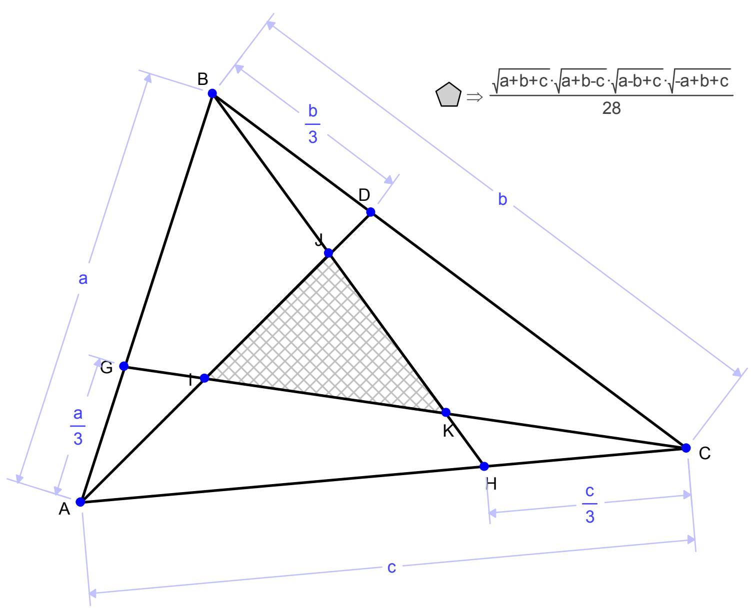

Figure 1: Feinman's Triangle

What if we use ¼ rather than 1/3?

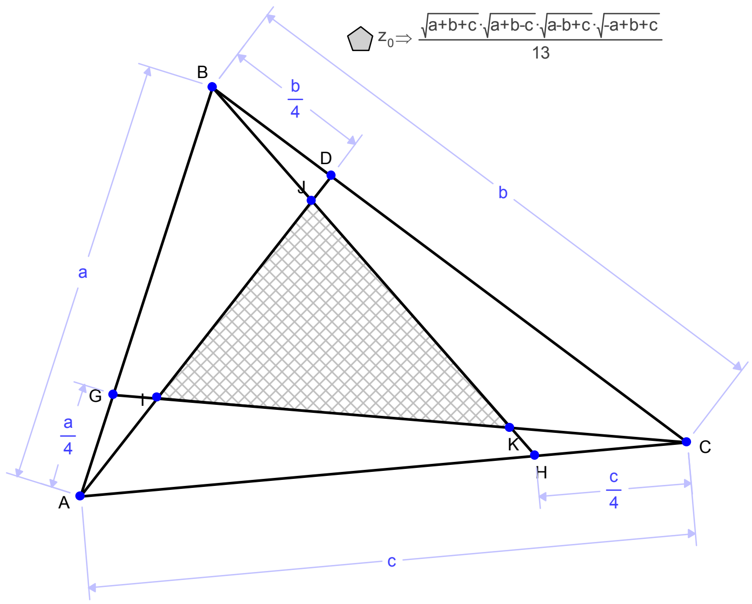

Figure 2: Triangle formed from the intersections of the lines dividing the sides of the triangle into quarters

We see that the area of the small triangle is 4/13 the area of the original triangle.

How about using 1/5?

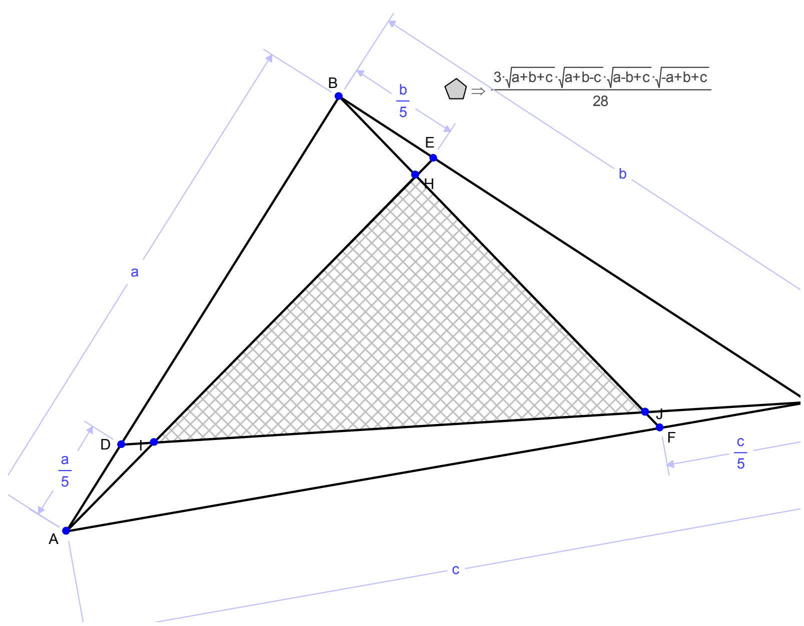

Figure 3: Triangle formed from the intersections of the lines dividing the sides of the triangle into fifths

Now we are 3/7 the area of the original.

In general we get this result:

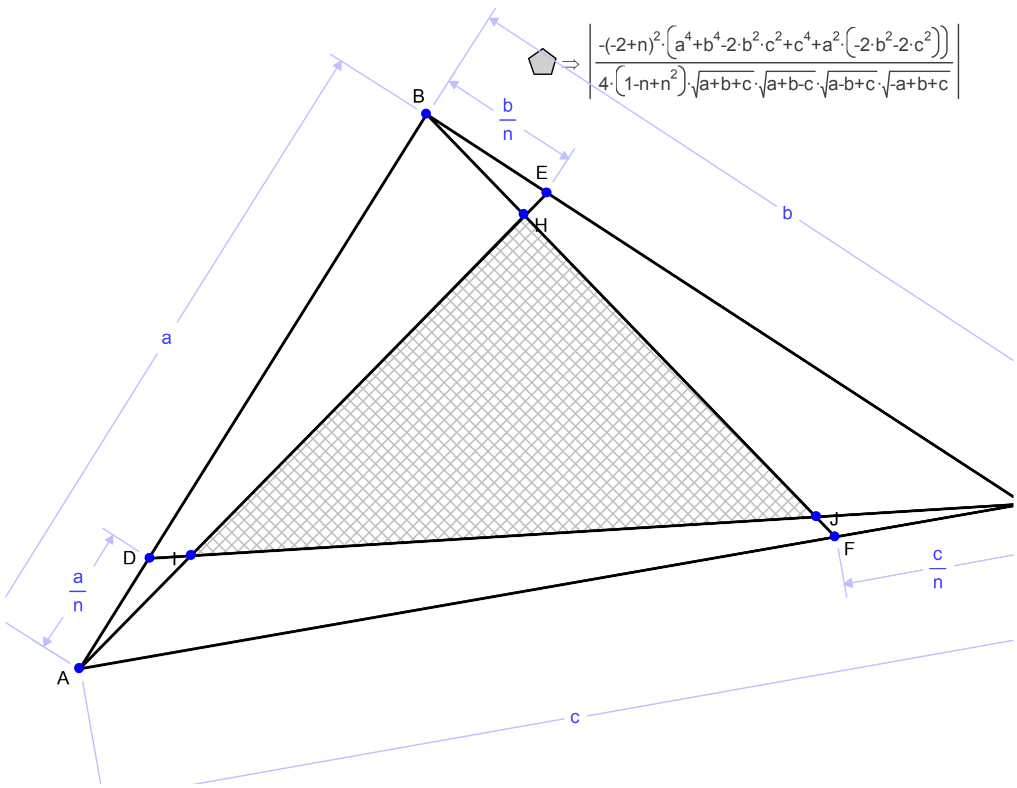

Figure 4: triangle formed by intersecting lines dividing sides into nths

We can copy into Maple and factor out the area of the original triangle::

```
> simplify(abs(-1/4*(factor(a^4+b^4-2*c^2*b^2+c^4+(-2*b^2-
2*c^2)*a^2))*(n-2)^2/(sqrt(-a+b+c)*sqrt(a-b+c)*sqrt(a+b-
c)*sqrt(a+b+c)*(-n+1+n^2))));
```

$$\frac{1}{4}\left|\frac{\sqrt{a+b-c}\,\sqrt{a+b+c}\,\sqrt{a-b+c}\,(n-2)^2\,\sqrt{-a+b+c}}{-n+1+n^2}\right|$$

Yielding an area ratio of:

$$\frac{(n-2)^2}{-n+1+n^2}$$

2. A Generalization of Feynman's Triangle

Instead of drawing our lines the same proportion along each side, what if we use different proportions:

For example:

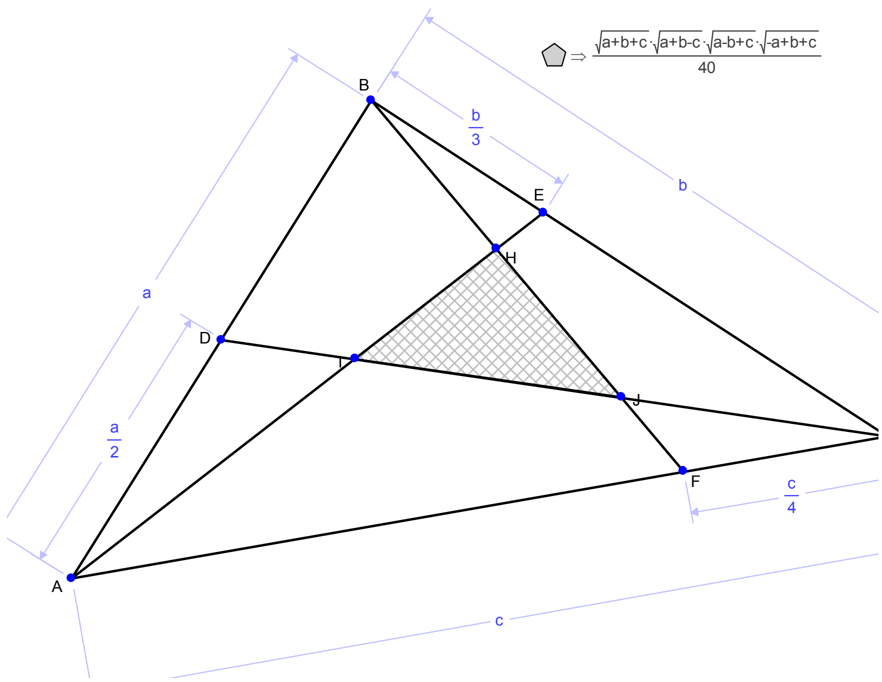

Figure 5: Dividing lines at different proportions of each side

1/2 , 1/3, and ¼ gives a triangle 1/10 of the original area.

The general result is shown in figure 6:

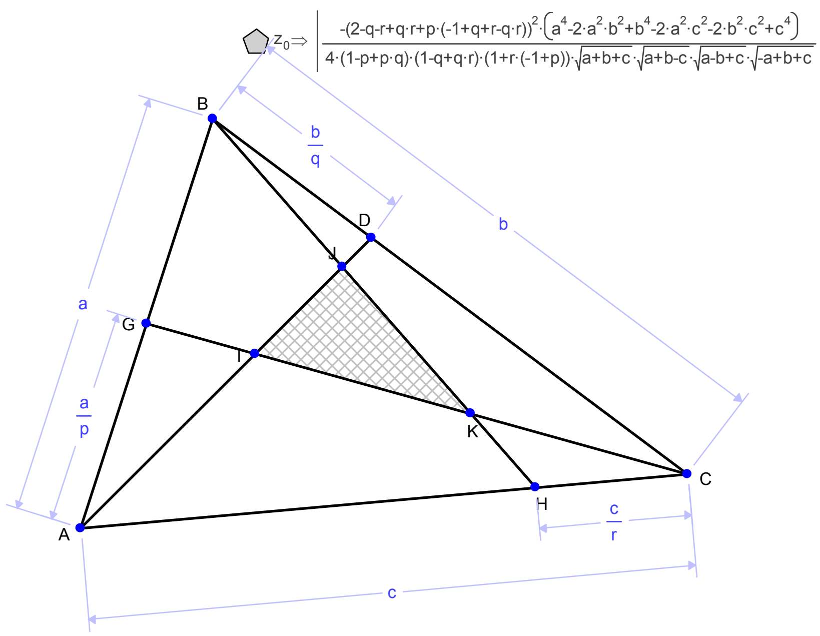

Figure 6: General form where lines divide the sides into p,q and r respectively

Copied into Maple and simplified (factoring the numerator) gives:

```
> simplify((-1/4*(factor(a^4-2*b^2*a^2+b^4-2*c^2*a^2-
2*c^2*b^2+c^4))*(-q+2-r+r*q+(-1+q+r-r*q)*p)^2/(sqrt(-
a+b+c)*sqrt(a-b+c)*sqrt(a+b-c)*sqrt(a+b+c)*((p-1)*r+1)*(1-
q+r*q)*(1-p+p*q))));
```

$$\frac{\sqrt{a+b+c}\,\sqrt{a+b-c}\,\sqrt{a-b+c}\,(-p\,r+p\,r\,q-p\,q+p+r-r\,q+q-2)^2\,\sqrt{-a+b+c}}{4\,(-r+p\,r+1)\,(1-q+r\,q)\,(1-p+p\,q)}$$

From which the area of the original triangle may be factored out to give the ratio of area of the small triangle to the original triangle as:

$$\frac{(-p\,r+p\,r\,q-p\,q+p+r-r\,q+q-2)^2}{(1-q+r\,q)\,(1-p+p\,q)\,(-r+p\,r+1)}$$

In general this evaluates to an unwieldy ratio. However, the values 4,7,8 are lucky and give a triangle exactly half the area of the original.

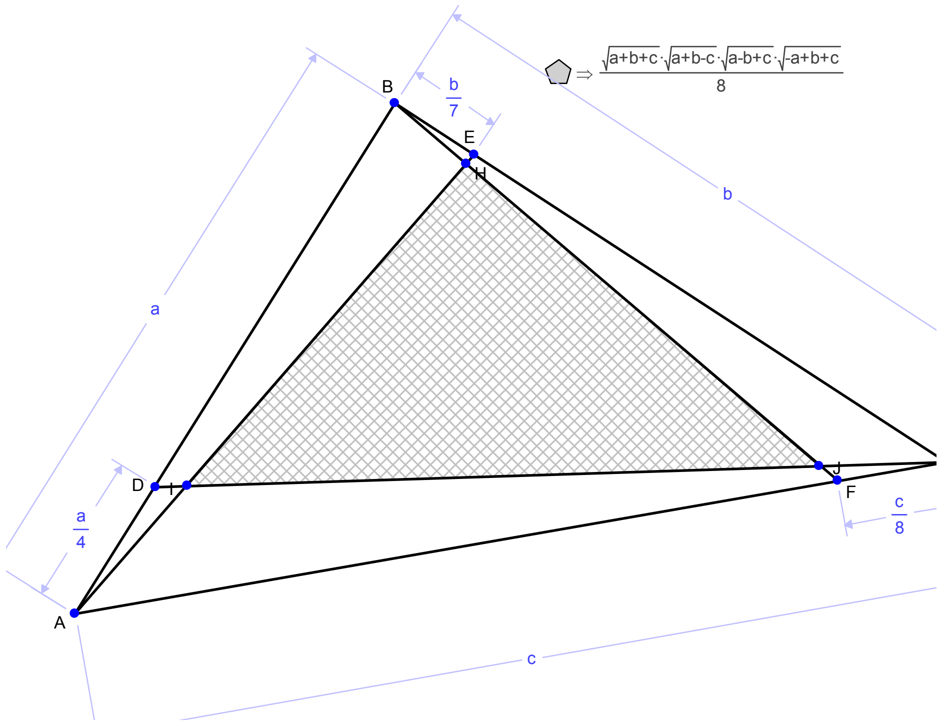

Figure 7: Using lines a quarter, a seventh and an eighth along the respective sides yields a triangle half the area of the original.

This general result was proved by Jacob Steiner a couple of centuries ago, and might be called Steiner's Theorem if it wasn't for the fact that there are no less than 3 other Steiner's Theorems