

Feynman Says: “Newton implies Kepler, No Calculus Needed!”

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Abstract: We recapitulate Feynman's demonstration that Kepler's laws follow from Newton's laws plus ordinary plane geometry: no calculus required. No differential equations, no conservation laws, no dynamics, no angular momentum, no constants of integration. This is Feynman at his best: reducing something seemingly big, complicated, and difficult to something small, simple, and easy.

1. Introduction

This presentation is very short, on purpose, but omits no detail. We back up the monologue with constructions in the software *Geometry Expressions*. To get a more lengthy presentation PLUS an audio CD recording of Professor Feynman delivering the lost lecture, see the book referenced at top. My method was to read the book, then put it away and reproduce the arguments from memory with my own twists tossed in. This is an original reproduction of Feynman's argument, as it were. Here are Kepler's three laws, in the order proved here (not Kepler's order):

K1: A planet orbiting a star sweeps out equal areas in equal times

K2: The square of the period of a closed orbit is proportional to the cube of its longest axis

K3: A closed orbit is an ellipse (open orbits are parabolic or hyperbolic)

For the sake of discussion, a “star” is defined as the fixed point in space about which the “planet” orbits. Bodies actually orbit around their mutual barycenter – their center of mass. There is a side theorem, not proved here, that the orbit of planet m and star M about their barycenter is equivalent to an orbit of planet m around a fictional fixed star of mass $M/(m+M)$ (the symmetric case also holds: the star can be thought of as orbiting a planet of tiny mass $m/(m+M)$, but that looks strange). For the following, presume the star is fixed in space.

Now, here are Newton's laws (specialized to the fixed-star assumption and numbered as we need them)

N1: A body (planet) in motion continues in straight-line motion unless acted on by a force

N2: The change in velocity of a planet over a time interval is proportional to the force applied (N1 is really a special case of N2 with force equal to zero)

N3: The force between a planet and a star acts along the line connecting them (*central-force law*)

N4: The magnitude of the gravitational force between a star and a planet is proportional to $1/R^2$, where R is the distance between them (*inverse-square law*)

Velocity is a vector: it has both magnitude – the *speed* – and direction. Force is also a vector, and to say that the change in a velocity is proportional to a force is to say that both magnitude and direction are proportional. There is a preliminary deduction we need to get out of the way: we only need plane geometry. The reasons are that if a planet starts out moving in a straight line and if any changes in its direction must be directed precisely toward the star, the planet can never leave the two-dimensional plane described by two points on the straight line of its motion and the one point located at the star.

Step 1: N1 and N3 imply K1 - equal areas in equal times

Notice we do not need the magnitude of the force – the inverse-square law, N4, is not needed for K1.

First, we prove the special case that if the force is zero, that is, if the planet proceeds in straight-line motion (N1), then equal areas are swept out in equal times (K1). Consider the following construction in *Geometry Expressions* (in file named Step1.gx):

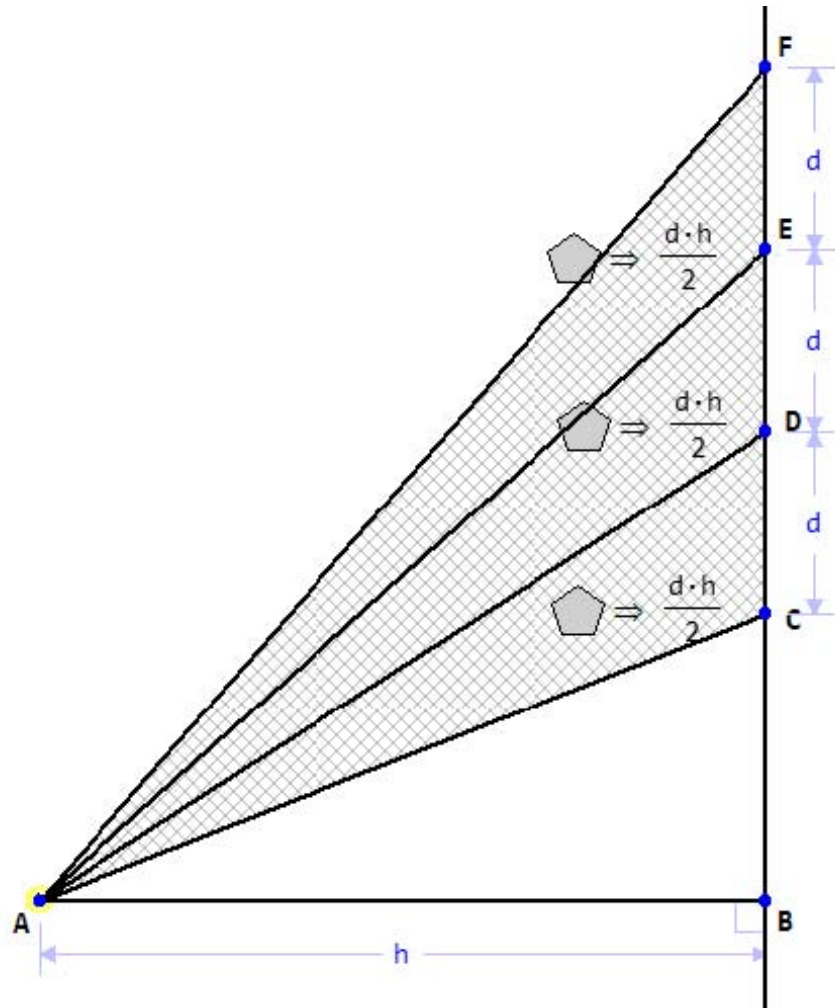


Figure 1. If the force is zero then equal areas are swept out in equal times

Let the star be at point A , and let the planet move with constant velocity along the vertical line BC . *Constant velocity* means two things: that in each equal interval of time, the planet moves a constant distance; and that the planet does not change direction. To express in *Geometry Expressions* that the planet moves constant distance in constant time, that is, that distances $CD=DE=EF$, constrain all those segments to be equal to the same variable, d , which is, conceptually, a constant velocity, v , times a constant small interval of time Δt . To express that the planet does not change direction, just make sure all the triangles have one leg coincident with line BC . In each time interval, the planetary path demarcates a triangle with the first point at the star, the second point at the beginning of the segment, and the third point at the end of the segment. Three such triangles are shown in the diagram, but, of course, there is a limitless number of them. All the triangles have exactly the same area. How do we know? They each have the same altitude, namely $h=AB$, and the same base, namely d . That is what we mean by "sweeping out equal areas in equal times." In the *.gx* file, move things

around. Make the triangles extra thin, extra fat, rotate them around: no matter what you do, their areas remain the same.

Now, let the star exert a little, percussive, attractive force on the planet – a tug – once each time step Δt , directed *precisely radially* toward the star. In the figure below, the tug is the vector, CE . By the parallelogram rule, $CE=DF$, and $CD=EF=d$. The original direction of the velocity, $CD=d=v\Delta t$, gets changed to the direction of CF through the action of the force. There is a big change to the direction and a small change to the magnitude:

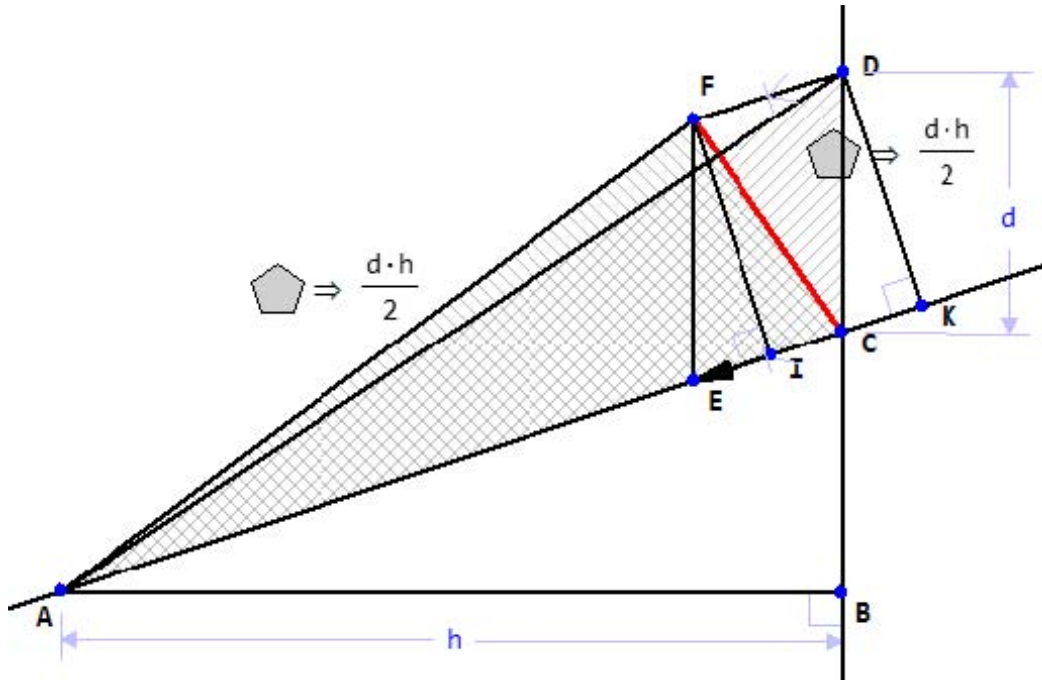


Figure 2. The star exerts a little percussive, attractive force on the planet

The really important thing is that the little tug force doesn't change the *area* of the original triangle, ACD ! The new triangle is ACF , which has base AC , just like the original, and altitude IF , which is equal to the altitude KD of the original triangle, by construction. Just to be sure we have everything right, we let *Geometry Expressions* calculate the areas of the two triangles. Sure enough, they're both $dh/2$. This is in the file *Step2.gx*. Again, move things all around the place and notice the areas cannot change.

Since the centrally-directed, *i.e. radial*, force does not change the area of this particular triangle, it can't change the area of *any* such triangle, since this one was not special. Just keep stitching together these tugged triangles, the way we stitched together the untugged triangles before, each starting at points like F above where the last triangle left off. Three steps of such an iterated construction appears in the next diagram (in file *Step7.gx*), with the jagged path of the planet, under the influence of repeated, percussive tugs from the star, in red. *Geometry Expressions* computes the six triangular areas symbolically and proves they are all equal despite the fact that I chose the magnitude of the

tugs capriciously. Only their directions matter. Radial forces never affect the area, even though they change the magnitude of the velocities a little, so the orbit sweeps equal angles in equal times under the action of *any* radial force. We have proved K1 using just N3 and N1.

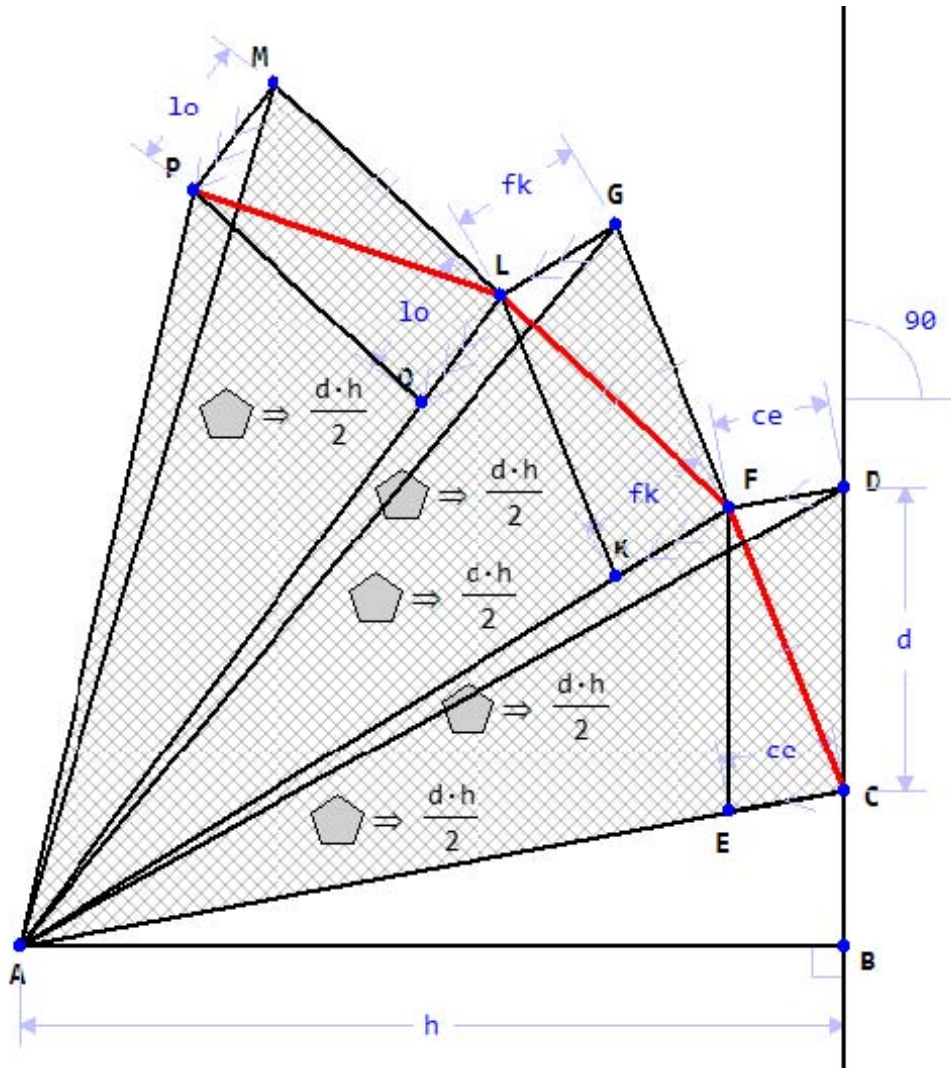


Figure 3. Radial forces never affect the area

Now, those who know calculus may be rolling their eyes and saying "ok, look, we can see through your subterfuge! All these little triangles, all these little forces, that little time interval Δt : you're obviously leading up to a limiting process, and you will cheat and drag in all of calculus." Not so. While it's true that a limiting process *would* lead to calculus, it's not necessary for these demonstrations. Everything is finite, not *infinitesimal*. It's true that things look better – more like real orbits – if the triangles are skinny and if the tugs and time intervals are small and closely spaced, but there is nothing in our logic that requires it, as playing around with the .gx files will show. Go ahead and drag everything around into grotesque shapes that don't look like parts of orbits: the areas will stay the

same, and that's all we set out to prove. In fact, the geometrical facts are *more general* than needed for the limiting process that would lead to the differential theory of orbits. It's precisely that latter theory, while more sophisticated and realistic, that is *not* needed to prove Kepler's laws, and that's the whole point of this demonstration. To drive the point home, file `Step11.gx` contains a complete, 12-point, discrete Keplerian orbit, and is illustrated below. The last leg, BA-C, is left free to avoid a lengthy constraint-resolution step. All the areas compute symbolically to $dh/2$, but only the first six are shown, to avoid lengthy algebraic computations.

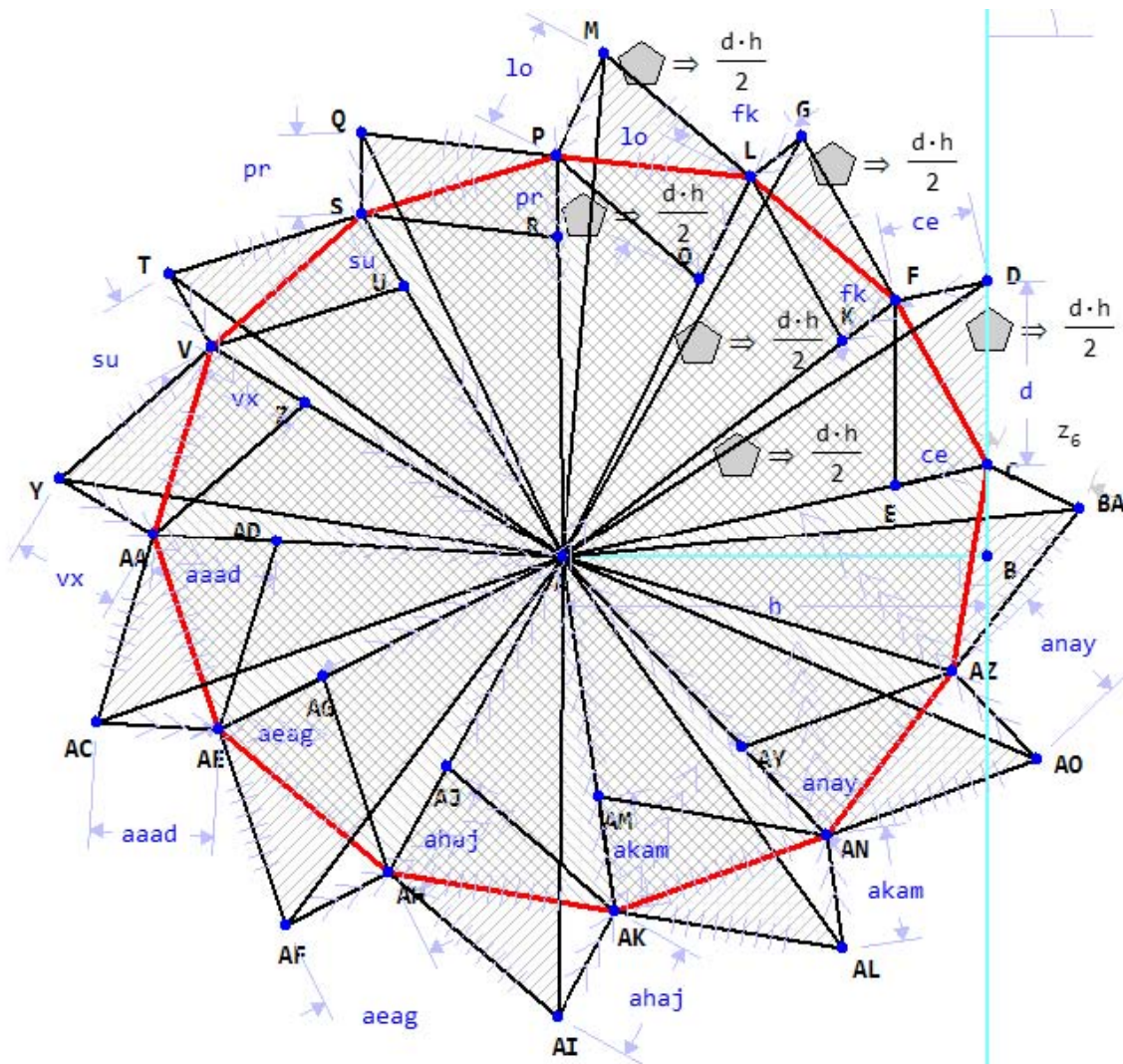


Figure 4. A complete, 12-point, discrete Keplerian orbit

Step 2: K2 - period squared proportional to radius cubed

This proof is easy for the special case of a **circular orbit** of radius R , where the planet's speed is also constant at every point. The distance the planet travels in one orbit is the circumference $2\pi R$. Let the period – the time it takes to orbit – be T . So, the average speed is $v=2\pi R/T$, which is also the speed at every point in the orbit.

The star's tugs on the planet don't change its speed, only the direction of its velocity. So, although the change in speed is zero, the change in velocity is not. How much does the velocity change? Imagine copying the velocity vectors into their own, abstract velocity space, rooting every vector at the origin – see "Diagram V" in figure 5 below. As the planet goes around the circle, the velocity vectors also sweep around their own circle – like a radar sweep. In this velocity circle, the velocity *changes* show up around the circumference. The *total* amount of velocity *change* in a single orbit is the circumference of the velocity circle, namely $2\pi v=2\pi^2 R/T$. Since this occurs over a time T , we may say that the average *rate* of change of velocity has magnitude $2\pi v/T=2\pi^2 R/T^2$. Since the orbit is circular and symmetrical, the average rate of change in velocity equals the rate of change in velocity at any time – it's just a constant. By N2, the rate of change in velocity is proportional to the force, which, by N4, is proportional to $1/R^2$. So we've shown that $R/T^2 \sim 1/R^2$, which is the same as saying $R^3 \sim T^2$.

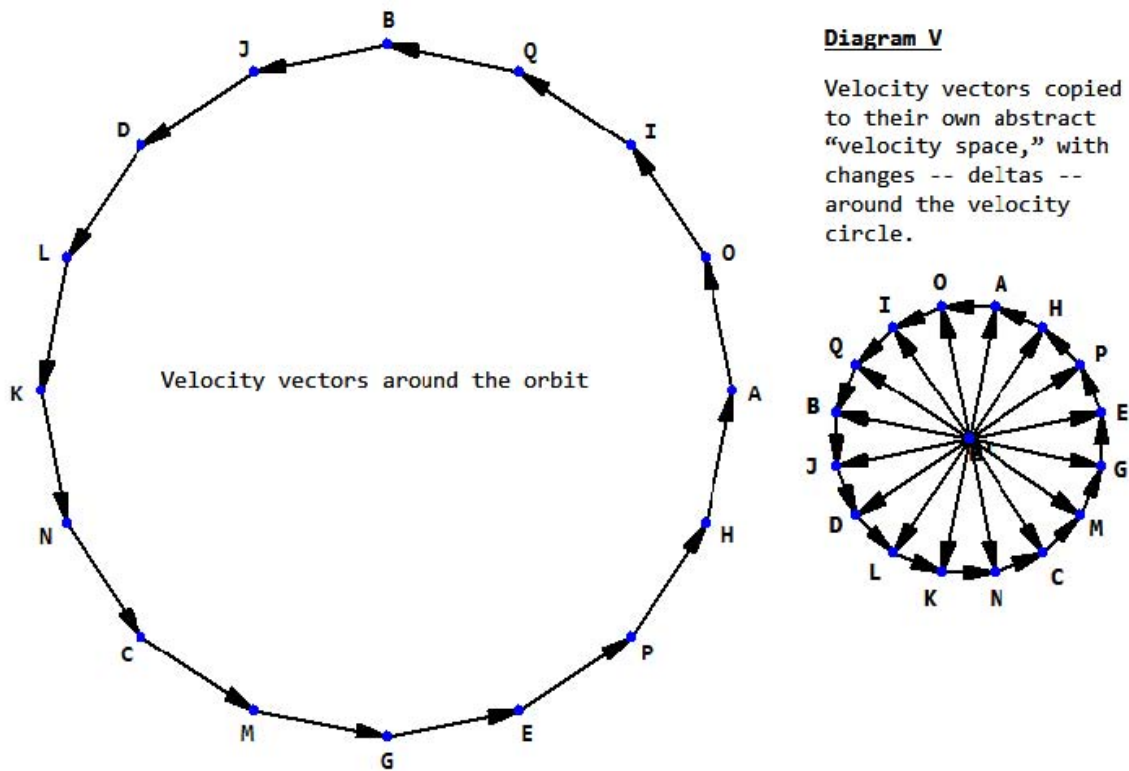


Figure 5. Velocity vectors around the orbit

Ok, so much for circular orbits. What about other shapes? Let's first find out what those other shapes are, then get back to proving K2 for them. K3 states that those other shapes are ellipses, so it's time to prove K3. Afterwards, we come back to prove K2 for ellipses.

Step 3: K3 – closed orbits are elliptical

We have seen that any radial force implies K1, equal areas in equal times. The magnitudes of the forces do not matter, only their directions. But arbitrary forces do not generate any particular shape unless we say something about the magnitudes of the forces. Time to bring in the inverse-square law.

Feynman observed that if we consider equal *angles* at the star instead of equal areas or equal times, each little tug will have identical magnitude and symmetric directions under N4, Newton's inverse-square law. Before proving these points, look at the following cartoon of an equiangular construction. This is deliberately cartoonish to emphasize the fact that we don't know the real shape of the orbit, yet:

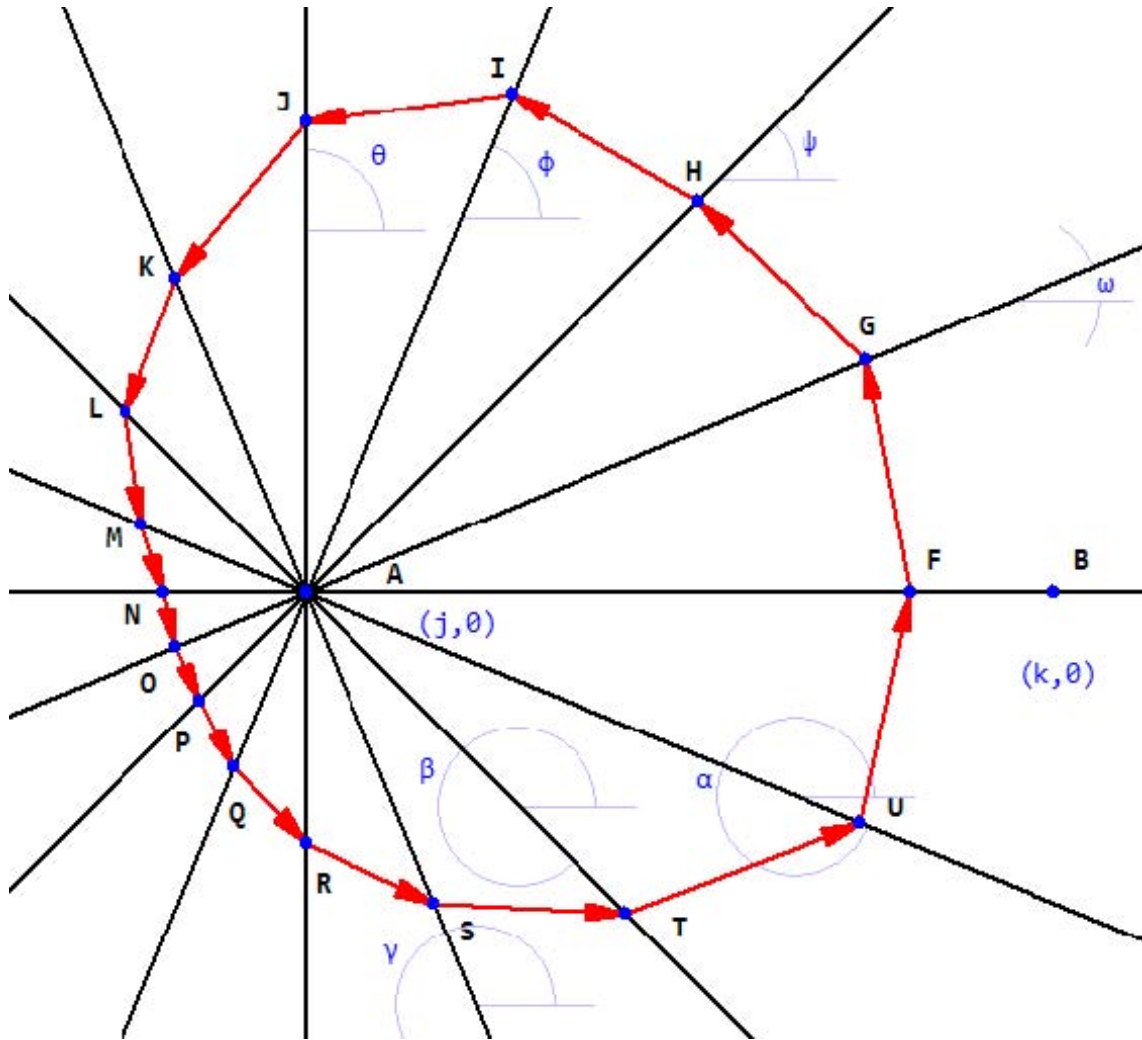


Figure 6. Deliberately cartoonish drawing of an equiangular construction

This is, again, Feynman at his absolute best: somehow finding just the right way to crack a problem. To understand his magic, observe that this is really a refinement of the prior, equal-time construction. Here's the argument:

First, realize that the area of any swept-out triangle of a given constant angle at the star is proportional to the squared distance from the star, that is, to R^2 . The reason is, roughly, that both the base and the height are proportional to R . More precisely, consider the following sketch (from [EquiAngular.gx](#)):

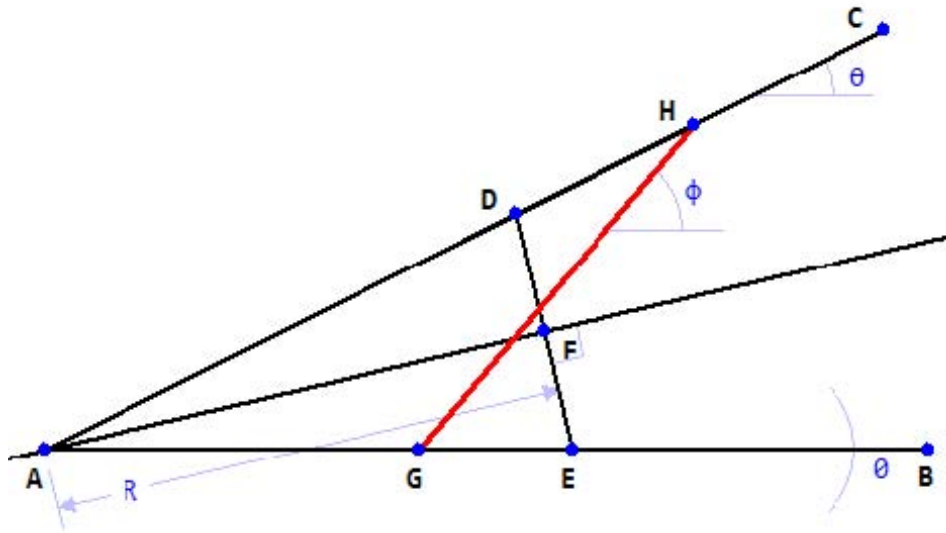


Figure 7. The area of any swept-out triangle of a given constant angle at the star is proportional to the squared distance from the star.

Let GH be the path of the planet, in red, as usual. It is always possible to find a point F a distance R from the star such that the area AGH equals the area of the triangle AED , which is constructed off the bisector of the angle θ at A with a perpendicular segment, ED , as a pair of right triangles, AFE and AFD . The important things are that this construction (1) gives us a way to define the "The Distance" R of the planet from the star while it is in segment GH of its orbit, and (2) makes it obvious that the area of the triangles is proportional to R^2 – the base, AF , is proportional to R and the height, DE , is also proportional to R .

The following figure shows two such triangles, AEF and AGH , laid on top of one another. Their angles at the star – point A – are equal; their "distances" from A are $AI \stackrel{\text{def}}{=} R_1$ and $AJ \stackrel{\text{def}}{=} R_2$, respectively. Points I and J are constructed specifically, as before, so that AEF and AGH have the same areas as the skew triangles, AKL and AMN , which contain segments of the orbit – KL and MN , respectively. The shaded triangle pairs, like $(PEL=PFK)$, and $(OGM=OHN)$ have equal areas, though they may not look so because the drawing is exaggerated to separate the points P from I and O from J .

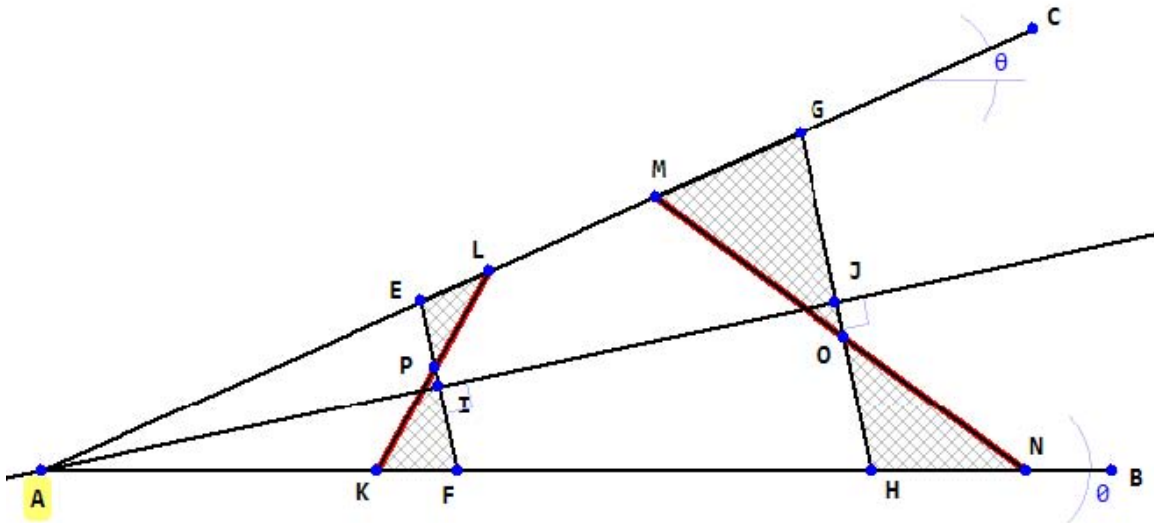


Figure 8. The area of the triangles is proportional to R^2

At this point, we've established that the areas of equiangular triangles in an orbit are proportional to the square distances of the planet from the star. We already know that the areas are proportional to the time intervals Δt , so they are no longer constant, but now proportional to R^2 ; symbolically, $\Delta t \sim R^2$. Recall N2 – *The change in velocity of a planet over a time interval is proportional to the force applied.* Write this symbolically as $\Delta v / \Delta t \sim F \sim \Delta v / R^2$. Recall N4 – *The magnitude of the force is proportional to $1/R^2$,* so N2 and N4 imply that $\Delta v / R^2 \sim 1/R^2$, that is, $\Delta v \sim 1$, that is, the magnitude of the change in velocity Δv is constant for each equiangular bit in an orbit.

The velocities change around the orbit, but the magnitudes of the velocity *deltas* are constant. What about the directions of the Δv changes? Since the orbit has been divided into equal angles and the velocity deltas – the tugs – are always radial, each tug varies from the previous by an equal angle. In other words, each Δv vector is of equal length and differs from its prior by an equal angle. The Δv 's go around the circumference of a circle in abstract velocity space, exactly like Figure 5 above. Only now, the velocities themselves do not necessarily radiate from the center of the velocity circle, but rather from some other, offset point. Of course, they all radiate from the same point, though, because that's how the abstract velocity space is set up. We have just deduced, however, given only N4 and the equiangular construction, that the velocities are constrained in exactly such a way that their deltas go around a perfect circle. The following, from Circle2.gx illustrates:

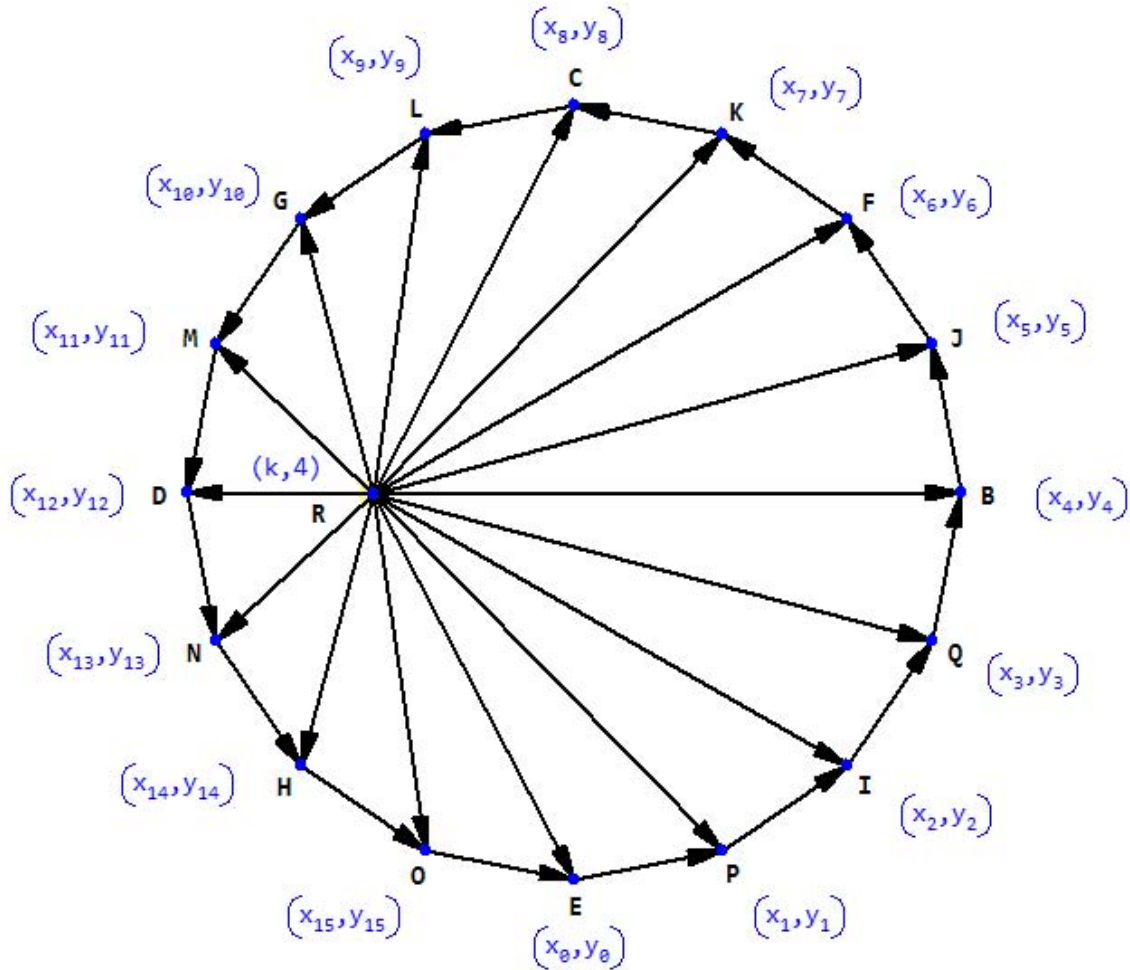


Figure 9. Velocities are constrained in exactly such a way that their deltas go around a circle.

In this file, point R is free to move horizontally, and by playing with it one can get a feel for exactly how the velocities are allowed and constrained to change in this construction. Since we know each velocity is tangent to the orbit, back in position space, not in abstract velocity space, we have a way to construct the orbit! We know the tangent intersects the orbit at exactly one point, so, if we can find where the tangent intersects the infinite line between the star and the planet, we've caught that one point! Take a deep breath and look at the following:

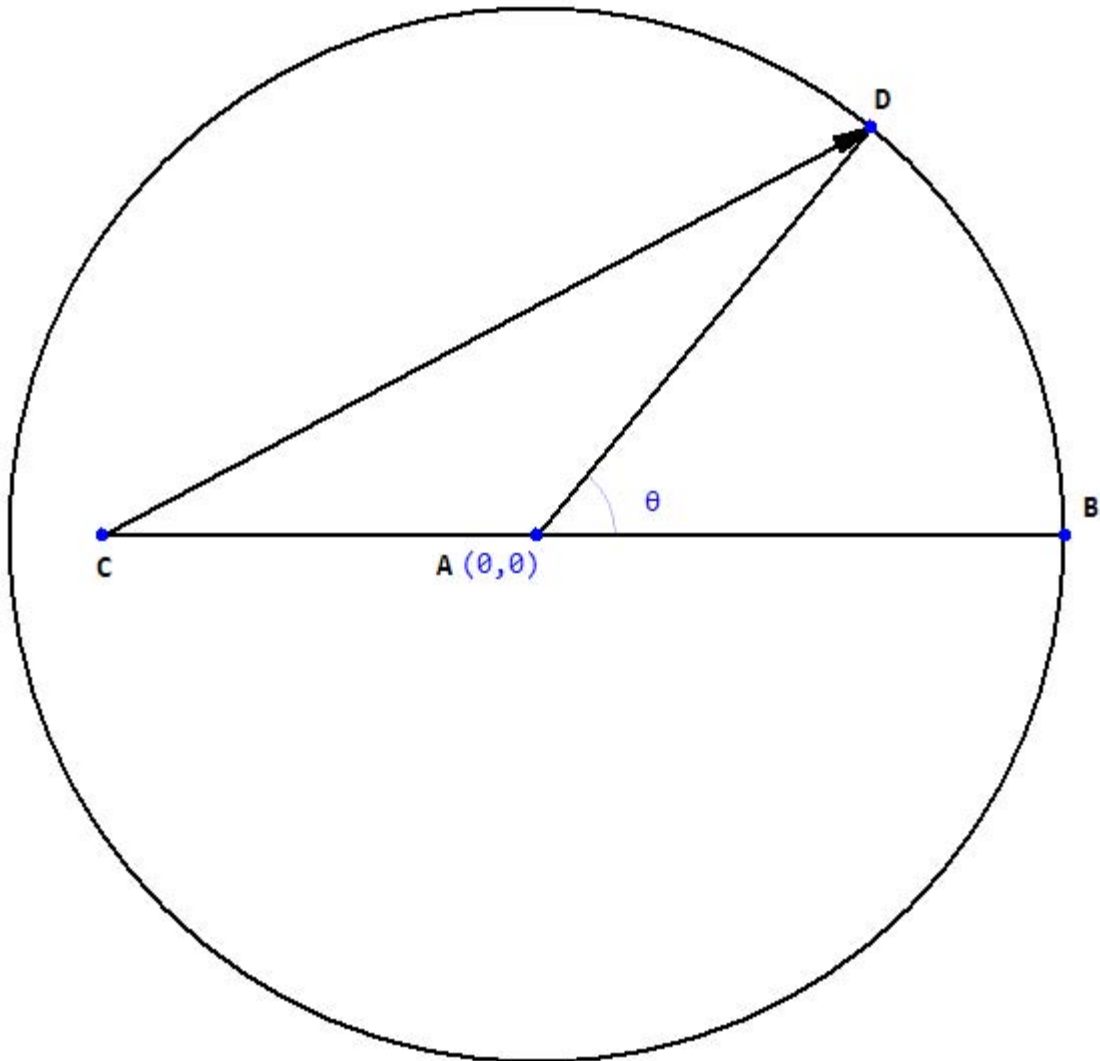


Figure 10. The total angle, θ , from the geometric center A of the velocity circle to D is equal to the total angle the planet has traveled around the orbit figure.

C is the velocity center of the velocity space, and CD shows both the magnitude and direction of a velocity. Since the circumference of the circle contains a number of equal, radial Δv 's, the total angle, θ , from the geometric center A of the velocity circle to D is equal to the total angle the planet has traveled around the orbit figure! This is perhaps the most difficult part of the entire demonstration to understand, but here is the way for you to convince yourself it is true. The largest velocity is CB , this should be plain to see. When the orbit has that largest velocity, the planet is closest to the star, because if the planet gets any closer, it will go faster because the force is greater the closer the planet gets and the force is proportional to $\Delta v/\Delta t$. Furthermore, the path of the planet must be perpendicular to the velocity at that closest point, because the planet is further from the star on either side of that closest point and the only way to do that is perpendicularly. Now, since the Δv 's accumulate by equal angles, let's say

1 degree at a time for a total of 360 little increments, and we set up the orbit to have equal angles to begin with, the increments of orbital angles must also be equal to one another. Once around the velocity circle for the Δv 's *in equal steps* corresponds to once around the orbit circle for the planet *in equal steps*. The angle at AD must stay in lock-sync with as the angle of the planet at the star. We might as well just consider the diagram above to be a combined velocity diagram and orbit diagram with AD being always ***perpendicular to*** a segment of the infinite line along the direction between the star and the planet. It's perpendicular to it because it starts out perpendicular when the velocity has its maximum value, CB , and it can never get out of sync, so it stays perpendicular always.

If we can find the intersection of the infinite direction line of the orbit with the infinite direction line of the velocity, then we have found a point on the orbit! Since we have the direction of the velocity, namely CD , and we have a perpendicular to the direction of the orbit, namely AD , we're almost there. The easiest thing to do is just take another perpendicular, this time to the velocity direction, at a proportional distance along its length. Two perpendiculars, one against the orbit direction and another against the velocity, make everything come out right! This is, again, Feynman at his unique best.

Take another deep breath and construct the perpendicular bisector, EF , representing a scaled version of the infinite direction line of the velocity. We don't care about the scale factor, because we're just interested in the shape of the orbit. Intersect EF with CD and we are guaranteed to have a point on a scaled and rotated version of the orbit. In *Geometry Expressions*, set up a locus construction for point E as angle θ varies from 0 to 360 and we have an ellipse. Proof? $CE=ED$ since triangles CEF and DEF are congruent. $AE+ED$ is constant since it's just the radius of a circle, therefore $AE+CE$ is constant, and that's the definition of an ellipse, just the old push-pins-and-string construction (in `Orbit1.gx`).

Reference

"Feynman's Lost Lecture, The Motion of Planets Around the Sun," by David L. Goodstein and Judith R. Goodstein, W. W. Norton & Company, New York, 1996.